Pure Altruism and Time Inconsistency: An Axiomatic Foundation

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Abstract

We study direct pure altruism of a generation towards its descendants, relying only on its preference over infinite consumption allocations. A generation exhibits *pure* altruism if it cares about its descendants' overall well-being—which may also incorporate their altruism—and *direct* altruism if it cares about the well-being of all its descendants directly. We show that direct pure altruism always leads to time inconsistency in the form of present bias. We obtain a new, tractable class of directly purely altruistic preferences that capture an impartial and coherent consideration of future generations. For this class, we study discounting of future generations' consumption utilities and how it depends on consumption levels. The only preferences that do *not* exhibit this dependence coincide with the quasi-hyperbolic discounting model, which our theory also characterizes. Finally, we examine how to conduct welfare analysis when the generations' preferences are time inconsistent.

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1 Introduction

When individuals face decisions whose consequences affect future generations, they often take such consequences into consideration. In other words, they exhibit intergenerational altruism. The importance of altruism has been recognized and adopted in economics to investigate a variety of problems.[1] By shaping the preferences of each generation, altruism influences how society evaluates and chooses between feasible courses of action. Although many models of intergenerational altruism have been developed and used, an investigation of the fundamental properties of preferences that exhibit altruism towards future generations seems to be missing—an exception, of course, is the standard model of exponentially discounted utility (EDU) (Samuelson (1937), Koopmans (1960)). Understanding the properties of those preferences is essential to assess merits and flaws of their different models. This paper aims to fill this gap, using a standard decision-theoretic approach.

We propose a general axiomatic foundation of direct pure altruism towards future generations. In our model, the present generation exhibits *pure* altruism because it derives utility—besides from its own consumption—from its descendants' overall utility (or well-being). Since it expects that future generations will continue to be altruistic, it ascribes to each of them a well-being that also takes account of their own utility from altruism.[2] Moreover, the present generation exhibits *direct* altruism because it cares about the well-being of all its descendants directly. Concretely, a grandmother cares about her son's well-being, understanding that it also depends on his daughter's well-being, but also directly about her granddaughter's well-being. Our analysis takes as primitive only the observable preference of the present generation over infinite consumption allocations to itself and future generations. Although this framework is standard, pure altruism raises conceptual as well as technical challenges. Our solution may prove useful for developing other models where similar preference interdependences across individuals arise.

In general, we characterize the class of preferences $\succ$ of the present generation (hereafter, generation 0) that have the following representation: For every stream $(c_0, c_1, \ldots)$, where c_t is the consumption of generation $t \geq 0$, generation 0's utility from c can be expressed as

$$U(c_0, c_1, \ldots) = V(c_0, U(c_1, c_2, \ldots), U(c_2, c_3, \ldots), \ldots). \qquad (1)$$

We interpret $U(c_t, c_{t+1}, \ldots)$ for $t > 0$ as the well-being that generation 0 *thinks* that

[1] See the discussion of the related literature in Section 2.

[2] Other adjectives have been used to denoted this type of altruism: nonpaternalistic (Ray (1987),Pearce (2008)) and total (Fels and Zeckhauser (2008)).

generation t derives from the then stream $(c_t, c_{t+1}, \ldots)$. In our model, generation 0's preference reveals that it altruistically cares about future generations and that, to assess their future well-being, it "projects" its own preference onto them.[3] A special case of representation (1) is, of course, Koopmans' (1960) model in which V does *not* depend on $U(c_t, c_{t+1}, \ldots)$ for $t > 1$. By contrast, direct pure altruism requires that V depend (positively) on $U(c_t, c_{t+1}, \ldots)$ for *all* $t > 1$. To distinguish the two cases, we shall refer to Koopmans' model as capturing *indirect* pure altruism (see also Section 3.2).

Direct pure altruism, alone, implies remarkable properties regarding the time consistency of intertemporal preferences. Imagine that all generations in a society have the same preference $\succ$ and that $\succ$ has a representation of the form (1) such that V is strictly increasing in all U's. Then the sequence of their preferences cannot be time consistent: A course of action that is optimal from the viewpoint of generation t need not be optimal, for that generation and its descendants, from the viewpoint of generation $t-1$.[4] Perhaps more surprisingly, each generation tends to be more impatient in the short run which directly involves itself, than in the long run which involves only future generations; in other words, we may say that each generation exhibits present bias. Intuitively, a grandmother can disagree with her son on whether he should transfer more consumption to his daughter, because they internalize differently the effects of changing her consumption: The son takes into account how her consumption affects his well-being through altruism, while in addition to this effect the grandmother also cares directly about her granddaughter's well-being. As a result, the grandmother may think that his son should transfer more consumption to his daughter than he wants to transfer. At the same time, however, if she were in his son's position, she would agree on keeping more consumption for herself. Thus she appears to be less impatient regarding delaying consumption when future generations are affected than when she is directly affected. Note that this property of the preferences is not assumed, but is rather a logical consequence of direct pure altruism.

A key contribution of the paper is to characterize more specific, but more tractable, forms of representation (1), which correspond to attractive properties—also from a normative viewpoint—of generation 0's preference. These representations take the additive form

$$U(c_0, c_1, \ldots) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U(c_t, c_{t+1}, \ldots)), \tag{2}$$

[3] Generation 0 may not know the preferences of generations who will be born many years into the future, or may see its preference as embodying some ethically desirable properties which future generations ought to adopt.

[4] For this conclusion to hold, it is enough that each generation cares about the well-being of its immediate descendant as well as some other future generation.

where $0 < \alpha < 1$ and the function G is strictly increasing and bounded. This function captures the altruism utility that generation 0 derives from future generations' well-being. The paper provides a further characterization of G that helps to appropriately choose it in applications, and derives a Bellman-type equation to compute consumption streams that maximize (2) for standard problems with dynamic resource constraints.

Besides some technical axioms, representation (2) relies on two key properties of generation 0's preference: *intergenerational separability* and *altruism stationarity*. Intuitively, intergenerational separability says that the well-being of future generations does not affect how generation 0 enjoys its consumption, and that the well-being of generation t does not influence how generation 0 ranks the well-being of any other generation s. As such, the first property also expresses a form of fairness or impartiality in the way generation 0 is altruistic towards future generations.

Altruism stationarity is the axiom, in our theory, which distinguishes direct from indirect pure altruism (i.e., our model from Koopmans' (1960) model and in particular EDU). Our stationarity notion focuses on the altruistic component of generation 0's preference by considering only changes in consumption streams that involve future generations. Intuitively, suppose that $c = (c_0, c_1, \ldots)$ and $c' = (c'_0, c'_1, \ldots)$ give the same consumption to a grandmother and that she thinks that her son is indifferent between $(c_1, c_2, \ldots)$ and $(c'_1, c'_2, \ldots)$ after he takes into account his consumption as well as the well-being of his daughter, granddaughter, and so on. Suppose that their well-beings induce the grandmother, who cares directly about them, to (strictly) prefer c to c'. Then, the axiom says that if the son were to die, his mother would continue to (strictly) prefer $(c_2, c_3, \ldots)$ to $(c'_2, c'_3, \ldots)$ for her granddaughter and so on. By requiring that each generation directly cares about future generations in a coherent way, altruism stationarity is also normatively appealing. As noted, this axiom fails for Koopmans' (1960) model. In this model, if the son is indifferent between $(c_1, c_2, \ldots)$ and $(c'_1, c'_2, \ldots)$, then his mother *must* also be indifferent, given her initial consumption. Once her son is dead, however, she may strictly prefer $(c_2, c_3, \ldots)$ to $(c'_2, c'_3, \ldots)$ for her granddaughter and so on.

Our derivation of representation (2) relies on known results in Debreu (1960) and Koopmans (1960) applied to the sequences of consumption of generation 0 and *well-beings* of the future generations that are implied by consumption streams. A complication arises, however, because the space of such sequences does not have a Cartesian-product structure, a property that plays an important role in the aforementioned papers. This is because the well-being of generation t depends on all its descendants' well-being. We identify a way to overcome this difficulty, which may be usefully employed in other settings where a decision maker cares about the well-being of other individuals in society and understands that they also care about each other's well-being.

Representation (2) implies several other properties of generation 0's preference. First, even though generation 0 cares directly about the well-being of all future generations, selfishness always dominates in the following sense: Faced with the choice between achieving higher satisfaction itself or granting more satisfaction to some future generation, generation 0 always chooses the former. Second, generation 0's well-being ultimately depends entirely on each generation's consumption utility (i.e., $u(c_0)$, $u(c_1)$, and so on). Thus we can examine how generation 0 trades off its own consumption utility against that of any future generation t. For a general G, this discount function turns out to depend on the entire consumption stream generation 0 is facing. This is because the well-being of generation t depends on the consumption of its descendants and affects the well-being of its ancestors. In particular, if G is concave (convex), then generation 0 discounts more (less) a stream that yields higher consumption utility to all future generations. Thus, for instance, after learning that future living standards will improve less than according to the historical trend, generation 0 may become more willing to sacrifice its own satisfaction for the good of future generations.

These consumption interdependences disappear when the function G in representation (2) is linear. Moreover, in this case discounting of consumption utilities takes a well-known, specific form: quasi-hyperbolic (or β-δ) discounting (Phelps and Pollak (1968), Laibson (1997)), where β and δ are simple functions of α and the constant slope of G. Linearity of G corresponds to an additional axiom, called *consumption independence*. This axiom says that (1) how a grandmother trades off her consumption with that of her son does not depend on the consumption—and hence well-being—of his descendants, and (2) how she trades off her consumption with that of her son's descendants does not depend on his consumption. Thus, as a byproduct of our general analysis, we provide an axiomatization of Phelps and Pollack's (1968) model of imperfect intergenerational altruism as a model of direct pure altruism. Perhaps ironically, Phelps and Pollak (1968) viewed the EDU model as capturing perfect intergenerational altruism, whereas from this paper's perspective EDU exhibits only indirect pure altruism. Thus, this paper offers an opposite view on which of these discounting models—the exponential and the quasi-hyperbolic one—captures a stronger degree of intergenerational altruism.

Finally, the paper addresses the delicate issue of how to conduct welfare analysis when direct pure altruism causes the generations' preferences to be time inconsistent. One implication of this causal relationship is to weaken the case for paternalistic interventions, which usually relies on viewing time inconsistency as a form of irrationality. In the case of Koopmans' (1960) model, social welfare is usually measured by the well-being of generation 0 (i.e., $U(c_0, c_1, \ldots)$). Our analysis suggests that this "libertarian" criterion seems more appropriate in the case of direct pure altruism—hence time *in*consistency—than in the case of indirect pure altruism—hence time consistency. This assessment is based on other properties of generation 0's preference, such as fairness and sensitivity towards future generations.[5]

One may argue that, despite generation 0's altruism, a social planner should aggregate the preferences of all generations, assigning an appropriate positive weight to each generation's well-being. We show that pure altruism—whether direct or indirect—makes it difficult to find an aggregator that also renders the planner time consistent.[6] One remarkable exception is representation (2) with linear G—hence the quasi-hyperbolic model. We show that the standard welfare criterion for β-δ preferences obtained by setting $\beta = 1$ corresponds to an aggregator that weighs the well-being of each generation t (including $t = 0$) by α^t, where α comes from representation (2) and satisfies $\alpha = \delta(1 - \beta)$.[7]

2 Related Literature

First of all, this paper is related to the vast literature on intergenerational altruism and its numerous applications. Examples include optimal national savings (Ramsey (1928), Phelps and Pollak (1968)), economic growth (Bernheim and Ray (1989)), charitable giving (Andreoni (1989)), family economics (Bergstrom (1995)), public finance (Barro (1974)), and environmental economics (Weitzman (1999), Dasgupta (2008), Schneider et al. (2012)). As Ray (1987) noted, "the representation of non-paternalistic functions [i.e., in terms of total utilities, U] in paternalistic form [i.e., in terms of consumption utilities, u] has been the subject of limited attention. A systematic analysis of the relationship between these two frameworks [...] appears to be quite a challenge, especially for models with an infinite horizon" (pp. 113–114). Saez-Marti and Weibull (2005) and Fels and Zeckhauser (2008) derive the mathematical equivalence between the β-δ formula

[5]Fairness and sensitivity to future generations have been proposed as desirable properties in the normative social-choice literature (see, for example, Asheim (2010) and the references therein). It seems not obvious that these properties are less important than time consistency.

[6]In his study of hedonistic altruism and welfare, Ray (2014) examines welfare criteria that aggregate well-being of altruistic, time-consistent, generations and that are formally similar to those in Section 5 (see also Bernheim (1989)).

[7]For other discussions on the "right" welfare criterion to use for the quasi-hyperbolic discounting model see, for example, O'Donoghue and Rabin (1999), Rubinstein (2003), Bernheim and Rangel (2009).

and expression (5) with linear G. Bergstrom (1999) studies systems of utility functions that include altruism towards others, focusing on the infinite regress that they may generate. All these papers, however, do not provide an axiomatic foundation of either representation. Arguably, although paternalistic forms of non-paternalistic functions may be more practical or explicit, to assess their merits and flaws we ultimately need to understand the fundamental properties—expressed in terms of axioms—of the preferences that those functions are meant to represent.

Many papers have proposed axiomatizations of the intertemporal preference of a decision maker—which makes it infeasible to have a complete list here. Koopmans (1960) derived a general model which contains as an important, special case Samuelson's (1937) EDU model. We will discuss the difference from Koopmans' analysis when we formally state our axioms. Other models that include quasi-hyperbolic discounting have been characterized by Hayashi (2003), Olea and Strzalecki (2014), and Echenique et al. (2014). We discuss how their approaches compare to ours in Section 3.3.

Time inconsistency of preferences (especially in the form of present bias) has been extensively studied and discussed since Strotz's (1955) seminal work, and has been given several explanations. In Akerlof (1991), present bias is based on a principle of cognitive psychology which says that decision-makers unduly overweigh relatively more salient or tangible events, such as present consumption relative to future one. In Gul and Pesendorfer (2001), time inconsistency can arise from a general change in the decision-maker's preference over time. In Halevy (2008) and Saito (2011), present bias can result from a combination of two things: (1) the present is usually certain, whereas the future is uncertain; (2) the decision-maker violates expected-utility theory and is disproportionately sensitive to certainty as in Allais (1953) and Kahneman and Tversky (1979). In Kőszegi and Szeidl (2012), time inconsistency and present bias arise because, when facing a present decision, the decision-maker focuses too much on its immediate consequences, but when considering that same decision ex ante, he is able to focus more on its overall consequences over time. Our theory provides a novel explanation for present bias, as part of a broader conceptual framework of how future consequences of current decision enter into the decision-maker's intertemporal preference.

Finally, this paper is related to the normative social-choice literature that studies how to rank intergenerational streams of consumption.[8] Our approach to this question partly differs from this literature's typical approach, which starts from some ethically desirable properties—possibly unrelated to people's real preferences—that rankings should satisfy

[8] See Asheim (2010) for a detailed review of this literature.

and derives their implied properties or functional representations. Desirable properties often suggested in this literature include sensitivity to and equal treatment of all future generations' well-being. The literature also highlights, however, that these properties collide with other standard, appealing properties such as completeness and continuity, which we maintain in the present analysis. Our contribution is to add the dimension of pure altruism to intergenerational preferences and to overcome the challenges that this addition creates: observability of future generations' well-beings and their interdependence. We also add a different perspective—through the notion of indirect pure altruism—on how Koopmans' (1960) model restricts the degree to which the present generation is sensitive to future generations' well-being. By contrast, preferences that exhibit direct pure altruism are sensitive to all future generations' well-being. Moreover, tractable versions of such preferences also exhibit an impartial and coherent treatment of future generations.

3 Preference Representations

3.1 Preliminaries

Consider a society that consists of an infinite sequence of generations, each indexed by a time period t with $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Each generation's consumption is denoted by c_t and belongs to a feasible set X, a connected, separable, metric space. The streams of consumption starting from generation 0, the one currently alive, are denoted by $c = (c_0, c_1, \ldots)$ and belong to the set $C = X^{\mathbb{N}}$. The set C is endowed with the sup-norm: $\|c - c'\|_C = \sup_t d(c_t, c'_t)$ where d is a bounded metric on X.[9] For $t \geq 1$, the set of consumption streams starting at t is ${}_tC \subseteq X^{\mathbb{N}}$—elements of this set will be denoted by ${}_tc = (c_t, c_{t+1}, \ldots)$.

The purpose of this paper is to study the preference of the present generation (hereafter, generation 0) over streams of consumption for itself and all future generations. To this end, we assume that generation 0 can commit to decisions (or policies) that determine such streams—for example, it may develop a new technology to produce clean energy that eliminates carbon emissions for the entire future. More formally, generation 0 can choose among all streams in C. We assume that these choices correspond to a well-defined, observable preference relation $\succ$ over C. Thus we interpret the expression $c \succ c'$ as saying that generation 0 deems c as a more preferable consumption stream for itself

[9] Assuming that $d(\cdot,\cdot)$ is bounded is without loss of generality, as we can always replace it by $\hat{d}(\cdot,\cdot) = d(\cdot,\cdot)/(1 + d(\cdot,\cdot))$, which is another metric respecting the initial distances.

and all future generations than c'. A basic premise of this paper is that $\succ$ has a utility representation. This is ensured by the following axioms (the symbols '$\succsim$' and '$\sim$' have the usual meaning).[10]

Axiom 1 (Weak Order)**.** *$\succsim$ is a complete and transitive binary relation.*

Axiom 2 (Continuity)**.** *For all $c \in C$, the sets $\{c' \in C : c' \precsim c\}$ and $\{c' \in C : c' \succsim c\}$ are closed.*

Axiom 3 (Future Constant-Flow Dominance)**.** *For all $c \in C$, there exist $x, y \in X$ such that $(c_0, x, x, \dots) \precsim c \precsim (c_0, y, y, \dots)$.*

Axioms 1 and 2 are standard. Axiom 3 captures the following intuitive idea: for any stream c, from generation 0's viewpoint there are consumptions x and y that are sufficiently bad and good, so that forcing each of the future generations to consume x (resp. y) is worse (resp. better) than forcing them to consume according to ${}_1c$. These axioms lead to the following standard result.[11]

Theorem 1 (Utility Representation)**.** *Under axioms 1-3, there exists a continuous function $U : C \to \mathbb{R}$ such that $c \succ c'$ if and only if $U(c) > U(c')$.*

In the rest of the paper, we will interpret $U(c)$ as the *well-being* (i.e., total utility) of generation 0 from stream c.

Since we are interested in studying the case of altruism towards future generations, by assumption the well-being of generation 0 depends on the consumption of some later generation. It is also natural that generation 0 cares about its own consumption.

Axiom 4 (Non-triviality)**.** *There exist $x, x', \hat{x} \in X$ and $c, c', \hat{c} \in C$ such that $(x, \hat{c}) \succ (x', \hat{c})$ and $(\hat{x}, c) \succ (\hat{x}, c')$.*

3.2 Pure-Altruism Representations

This paper aims to investigate which properties of $\succ$ correspond to pure (or nonpaternalistic) altruism of generation 0 towards future generations. Intuitively, generation 0 exhibits pure altruism if it derives utility from the total utilities—or well-being in our

[10] This paper continues to assume, as in the EDU model, that the preference of generation 0 does not depend on the consumption of past generations. Relaxing this assumption, though interesting, is beyond the paper's scope.

[11] The proofs of the main results are in the Appendix A. All omitted proofs are in Appendix B (Online Appendix).

terminology—of future generations (see, e.g., Ray (1987)).[12] Since $\succ$ corresponds to the subjective attitude of generation 0 towards future generations, $\succ$ can only reveal how this generation perceives the well-being each future generation t derives from the then stream ${}_tc$. However, generation 0 may not know future generations' preferences—for instance, for those born many years into the future—or it may have no reason to believe that their preference will differ systematically from its own—for instance, because it views this preference as based on some generally appealing and sensible ethical norms. In these cases, generation 0 may simply "project" its preference onto future generations and use it to assess their well-being. This is the approach that we take in this paper.[13]

This leads to the following general class of representations of purely altruistic preferences. Fixing a representation U of $\succ$, let $\mathcal{U}$ be the range of U and define

$$\mathcal{F} = \{(f_1(c), f_2(c), \ldots) : f_t(c) = U({}_tc) \text{ for } c \in C \text{ and } t > 0\}. \tag{3}$$

Note that $\mathcal{F} \subset \mathcal{U}^{\mathbb{N}}$, but in general $\mathcal{F}$ is not a Cartesian product (e.g., $\mathcal{F} \neq \mathcal{U}^{\mathbb{N}}$) because the well-being from ${}_tc$ depends on future consumption and hence on future well-being.

Definition 1 (Pure-Altruism Representation)**.** Preference $\succ$ has a pure-altruism representation if and only if

$$U(c) = V(c_0, U({}_1c), U({}_2c), \ldots) \tag{4}$$

for some function $V : X \times \mathcal{F} \to \mathbb{R}$ that is nonconstant in c_0 and $U({}_tc)$ for some $t > 0$.

Thus how generation 0 ranks streams c and c' depends only on its own consumption c_0 and c'_0, and on how it perceives—through the lenses of its current preference—that some future generation t will rank the continuation streams ${}_tc$ and ${}_tc'$.

Axiom 5 below is the key to obtaining representation (4). It captures a minimal property that seems natural for purely altruistic preferences: given its own consumption, if generation 0 *thinks* that two consumption streams will render all future generations indifferent, it should also be indifferent.

Axiom 5. *If* ${}_tc \sim {}_tc'$ *for all* $t > 0$*, then* $(c_0, {}_1c) \sim (c_0, {}_1c')$.

The premise ${}_tc \sim {}_tc'$ captures the thought experiment of generation 0 which imagines to face the same continuation streams that generation t will face and uses its current

[12]By contrast, generation 0 is impurely (or paternalistically) altruistic if it cares only about the utility that other generations derive from what they actually consume. To the extent that generation 0 thinks that future generations are also altruistic, this utility from actual consumption differs from each generation's total utility.

[13]For another axiomatic model of altruism in which the decision-maker projects its preference onto his peers, see Saito (2013) for example.

preference to assess how that generation will rank ${}_tc$ and ${}_tc'$. Axiom 5 rules out the possibility, for instance, that generation 0 prefers c to c' because, despite generating the same stream of current consumption and future well-being, they allocate future consumption differently across generations.

Theorem 2. *Axioms 1–5 hold if and only if $\succ$ has a pure-altruism representation.*

Proof. Define $\mathcal{F}_0 = X \times \mathcal{F}$ and let $f_0(c) = c_0$ and $f(c) = (f_0, f_1, f_2, \ldots)$.

($\Rightarrow$) First, we define equivalence classes on C as follows: c is equivalent to c' if $f_t(c) = f_t(c')$ for all $t \geq 0$.[14] Let C^* be the set of equivalence classes of C, and let the function U^* be defined by U on C^*. Then, the function $f^* : C^* \to \mathcal{F}$, defined by $f^*(c^*) = f(c)$ for c in the equivalence class c^*, is by construction one-to-one and onto. Let $(f^*)^{-1}$ denote its inverse and, for any $f \in \mathcal{F}_0$, define

$$V(f) = U^*((f^*)^{-1}(f)).$$

By Axiom 5, V is a well-defined function, and $V(f(c)) = U(c)$ for every c. By Axiom 4, V is nonconstant in f_0 and f_t for some $t > 0$.

($\Leftarrow$) Suppose that $V : \mathcal{F}_0 \to \mathbb{R}$ is a function such that $V(f(c)) = U(c)$ and V is nonconstant in f_0 and f_t for some $t > 0$. Then, it is immediate to see that the implied preference satisfies Axioms 4 and 5. □

We emphasize that Axiom 5 is weak: it requires that generation 0 be indifferent between two streams, only if it perceives that *all* future generations—not just generation 1—will be indifferent between their continuation streams. Clearly, if generation 0 cares only about the well-being of generation 1—as in the EDU model—Axiom 5 holds. By allowing the preference of generation 0 to depend on the (perceived) well-being of future generations in a richer way, Axiom 5 is a key step of our approach to modeling intergenerational preferences differently from EDU (Koopmans (1960, 1964)).

To distinguish our approach from the standard one, we introduce the following terminology. As noted, EDU satisfies

$$U(c) = u(c_0) + \delta U({}_1c) = V(c_0, U({}_1c)).$$

In this case, generation 0 is purely altruistic only towards generation 1, and by expecting that all future generations will be altruistic in a similar way, it ends up caring about how c affects all of them but only indirectly through the well-being of generation 1. For

[14] In general, there may be several consumption streams in an equivalence class. For example, suppose that $U(c) = c_0 + c_1 + c_2 + c_3$, and let $c = (1, 1, -1, -1, 1, 1, -1, -1, \ldots)$ and $c' = (1, -1, -1, 1, 1, -1, -1, 1, 1, \ldots)$.

this reason, when generation 0's preference $\succ$ has a representation of the form $U(c) = V(c_0, U({}_1c))$, we shall say that generation 0 exhibits *indirect pure altruism*.[15] On the other hand, we shall say that generation 0 exhibits *direct pure altruism* if its well-being $U(c)$ depends directly on the well-being of all future generations (i.e., V in (4) depends on $U({}_tc)$ for all $t > 0$).[16]

3.2.1 Time (In)consistency and Present Bias

It is well known that if each generation has the same preference and this preference can be represented by the EDU model—implying indirect pure altruism—then their preferences are time consistent. That is, if a course of action starting at time t is preferable according to generation t's preference, then it remains preferable, *for time* t, according to generation $t-1$'s preference. To formalize this, we introduce a family of preferences $\{\succ^t\}_{t=0}^{\infty}$, where $\succ^t$ is the preference relation of generation t. We then have the following definition (see, e.g., Siniscalchi (2011)).

Definition 2 (Time Consistency)**.** Preferences $\succ^{t-1}$ and $\succ^t$ satisfy time consistency if the following condition holds: ${}_tc \succsim^t {}_tc'$ if and only if $(c_{t-1}, {}_tc) \succsim^{t-1} (c_{t-1}, {}_tc')$.

Proposition 1 below shows that direct pure-altruism is incompatible with time consistency. The purpose of this preliminary result is simply to identify and highlight a possible source of time inconsistency across generations. This source corresponds to the intuitive idea that each generation directly cares about the well-being of future generations beyond its immediate descendant. Concretely, a grandmother usually cares about her son's well-being—which of course depends on his daughter's well-being—and *directly* about her granddaughter's well-being. By contrast, in the EDU model, it is as if the grandmother cares about her granddaughter's well-being only *indirectly* to the extent that it affects her son's well-being.[17]

Proposition 1. *Consider a family of preferences $\{\succ^t\}_{t=0}^{\infty}$ and suppose that $\succ^t = \succ^0$ for all $t > 0$ and that $\succ^0$ has a pure-altruism representation. Then, $\{\succ^t\}_{t=0}^{\infty}$ satisfies time consistency if and only if*

$$V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c))$$

[15] Another possible interpretation is that EDU captures a generation 0 which fails to realize that future generations will continue to be altruistic and to care about the well-being of their descendants.

[16] Of course, one could also consider the case in which V depends on $U({}_tc)$ up to some finite $\bar{t} > 1$.

[17] The purpose of Proposition 1 should not be misunderstood. It is well known that a family $\{\succ^t\}_{t=0}^{\infty}$ satisfies time consistency if and only if $\succ^t$ has a specific recursive representation in which $U({}_tc)$ depends only on current consumption, c_t, and continuation utility $U({}_{t+1}c)$.

for all $c \in C$, and V is strictly increasing in its second argument.

It is common to view time consistency of intertemporal preferences as the norm and time inconsistency as an exception. Proposition 1 reverses this view. If we deem natural that each altruistic generation cares about future generations' well-being beyond the immediate future and expects them to do the same, then we have to conclude that time inconsistency should be the norm, not the exception.

To see the intuition behind Proposition 1, suppose that ${}_1c$ and ${}_1c'$ represent two courses of actions that start in period 1 and involve different consumptions only for generations 1 and 2 (i.e., $c_1 \neq c'_1$, $c_2 \neq c'_2$, but ${}_3c = {}_3c'$). In this case, generation 0 (the grandmother) and generation 1 (her son) may disagree on the ranking of ${}_1c$ and ${}_1c'$ for the following reason. The son trades off how its well-being varies because his current consumption changes from c_1 to c'_1 and how it varies because his daughter's well-being is affected by the change from c_2 to c'_2. In addition to these two effects on her son's well-being, the grandmother also takes *directly* into account how her granddaughter's well-being varies between c_2 and c'_2. Therefore, the grandmother and her son internalize the effects of changing c_2 in different ways, which can induce them to disagree on which course of action is better.

In general, time inconsistency can take different forms. For instance, if we allow the preferences to differ in arbitrary ways across generations, we could have that generation t prefers higher consumption (say, of fossil fuels) for itself and all future generations, whereas generation $t+1$ prefers lower consumption for itself and all subsequent generations. This kind of inconsistency is clearly possible, but differs from the one arising here. In particular, directly purely-altruistic preferences are time inconsistent even if *all* generations have the same preference relation. Moreover, this inconsistency always takes a very specific and well-known form: each generation tends to be more impatient in the short run than in the long run. To be consistent with the literature, we shall call this phenomenon "present bias."

Definition 3 (Present Bias)**.** Let x, y, w, $h \in X$ and $c' \in C$. Suppose that $(x, c) \succ (y, c)$ for all c and $(z_0, \ldots, z_{t-1}, x, w, c') \sim (z_0, \ldots, z_{t-1}, y, h, c')$ for some $t > 0$, then $(x, w, c') \succ (y, h, c')$.

Intuitively, this definition says the following. Suppose first that, fixing consumption for all future generations, generation 0 always strictly prefers consumption x to y. We can thus unambiguously say that generation 0 likes x better than y for itself and all future generations. Now suppose that there is a consumption h such that generation 0 is indifferent if generation t is forced to give up x for y, provided that this change is "compensated" by giving h to generation $t+1$. Then, if generation 0 itself faced the choice between x and y in the present, it would strictly prefer *not* to give up x for y even if in exchange generation 1 got h. Thus, generation 0 pursues its current gratification.

Proposition 2. *If $V(c_0, U(_1c), U(_2c), \ldots)$ is strictly increasing in $U(_tc)$ for all $t > 0$, then $\succ$ exhibits present bias.*

As shown in the proof of Proposition 2, its conclusion continues to hold if in Definition 3 both w and h occur at some periods s after $t+1$.

Though perhaps surprising, this result has a simple intuition. Set $t = 1$ in Definition 3 and imagine that x and y correspond to higher and lower consumption. From the viewpoint of a grandmother (generation 0), a reduction in her son's consumption from x to y can be compensated by an increase in her granddaughter's consumption, h, that is large enough to sufficiently increase her granddaughter's well-being as well as her son's well-being through his altruism towards his daughter. However, because the grandmother takes directly into account her granddaughter's well-being, the level h that renders her indifferent may be insufficient to render her son indifferent. This implies that if the grandmother were in the same situation of her son, she would also strictly prefer not to reduce her own consumption from x to y even if her child received h. Thus, the grandmother appears to be less impatient with respect to delaying consumption when future generations are affected than when she is directly affected.

3.3 Time-separable, Stationary Preferences

To obtain sharper results, we now refine the general representation V in (4), by considering preferences that satisfy some form of intergenerational separability and stationarity. These properties will also yield tractability. In this section, we go back to considering only one single preference, namely that of generation 0.

The first axiom captures the idea that $\succ$ is separable across generations—that is, separable between generation 0's consumption and future generations' well-being, as well as across their well-being. Intuitively, this means that the well-being of future generations does not affect how generation 0 enjoys its own consumption. Moreover, the well-being of generation t does not influence how generation 0 ranks the well-being of any other generation s. As such, this notion of separability captures a minimal, appealing degree of fairness in the way generation 0 is altruistic towards future generations. To state Axiom 6, let Π consist of all unions of subsets of $\{\{1\}, \{2\}, \{3, 4, \ldots\}\}$.

Axiom 6 (Intergenerational Separability). *Fix any* $\pi \in \Pi$*. If* c*,* $\hat{c}$*,* c'*,* $\hat{c}' \in C$ *satisfy*
(i) ${}_t c \sim {}_t \hat{c}$ *and* ${}_t c' \sim {}_t \hat{c}'$ *for all* $t \in \pi$*,*
(ii) ${}_t c \sim {}_t c'$ *and* ${}_t \hat{c} \sim {}_t \hat{c}'$ *for all* $t \in \mathbb{N} \setminus \pi$*,*
(iii) either $c_0 = c'_0$ *and* $\hat{c}_0 = \hat{c}'_0$*, or* $c_0 = \hat{c}_0$ *and* $c'_0 = \hat{c}'_0$*,*
then $c \succ c'$ *if and only if* $\hat{c} \succ \hat{c}'$*.*

To illustrate Axiom 6, consider $\pi = \{1\}$. Suppose that c and c' give the same consumption to generation 0 and, according to its viewpoint, yield the same well-being for all future generations except generation 1. Suppose that, due to generation's 1 different well-being, generation 0 prefers c to c'. Now, consider changing c and c' (to $\hat{c}$ and $\hat{c}'$) in any way so that the well-being that generation 0 ascribes to generation 1 does not change, while generation 0's consumption and the well-being that it ascribes to all other generations changes in the same way. According to the axiom, generation 0 should prefer $\hat{c}$ to $\hat{c}'$, thereby continuing to let the well-being of generation 1 determine its current ranking. At a formal level, Axiom 6 is inspired by Debreu's (1960) and Koopmans' (1960) separability axioms. It differs, however, at a substantive level in requiring that certain consumption streams be indifferent from generation 0's viewpoint, rather than that certain physical consumptions be equal. This is because we want separability in future generation's well-being as perceived by generation 0.

Axiom 7 captures some weak and natural monotonicity property of altruistic preferences. First, everything else equal, generation 0 is better off if it thinks that generation 1 will be better off. Second, if for *any* horizon T and *any* common consumption stream of the generations after T generation 0 prefers the consumption of the first T generations implied by c to that implied by c', then it also prefers the whole stream c to c'. Intuitively, the second property rules out the possibility that the well-being ascribed to a generation in the infinite future could overturn how generation 0 ranks c and c', even though, having fixed that generation's well-being, generation 0 thinks that all intermediate generations are better off with c than with c'.

Axiom 7 (Monotonicity). *Let* c *and* c' *be any stream in* C*.*
(i) If $c_0 = c'_0$*,* ${}_1 c \succ {}_1 c'$*, and* ${}_t c \sim {}_t c'$ *for all* $t > 1$*, then* $c \succ c'$*.*
(ii) If for any T *and continuation stream* c'' *we have* $(c_0, c_1, ..., c_T, c'') \succsim (c'_0, c'_1, ..., c'_T, c'')$*, then* $c \succsim c'$ *.*

Two Notions of Stationarity

All properties introduced so far (Axioms 1-7) are also satisfied by Koopmans' (1960) standard model of intertemporal preferences—i.e., by a representation of the form $U(c) =$

$\hat{V}(u(c_0), U(_1c))$ with $\hat{V}$ strictly increasing in each argument, and hence by the EDU model. Thus, both indirectly and directly purely-altruistic preferences satisfy these axioms. It turns out that it is possible to distinguish these kinds of altruism in terms of one single additional property, called stationarity. Adding Koopmans' (1960) notion of stationarity to Axioms 1-7 completes the characterization of indirectly purely-altruistic preferences, by delivering representation $U(c) = \hat{V}(u(c_0), U(_1c))$. We discuss this notion at the end of this section and explain why it appears problematic to us.

Our new notion of stationarity takes the theory to a different path and characterizes direct pure altruism. This notion focuses on the altruistic component of generation 0's preference by considering only changes in consumption streams that involve future generations, but leave its current consumption unchanged. Intuitively, if generation 0 cares *directly* about the well-being of generations beyond its immediate descendant in a coherent way, it should be possible to "remove" generation 1 and preserve how generation 0 ranks the consumption streams starting from generation 2 onward. Clearly, for this comparison to be meaningful, we must start from a situation in which how generation 0 ranks two streams in the presence of generation 1 depends only on the well-being that generation 0 ascribes to subsequent generations but not on the well-being of generation 1. By requiring that grandparents directly cares about the well-being of their grandchildren, great grandchildren, and so on in a coherent way, Axiom 8 also has a natural normative appeal.

Axiom 8 (Altruism Stationarity)**.** *If* $c, c' \in C$ *satisfy* $c_0 = c'_0$ *and* $_1c \sim {_1c'}$*, then*

$$c \succsim c' \Leftrightarrow (c_0, {_2c}) \succsim (c'_0, {_2c'}).$$

To gain intuition for Axiom 8, suppose that c and c' give the same consumption to a grandmother and that she thinks that her son is overall indifferent between $_1c$ and $_1c'$ after he takes into account his consumption as well as the well-being of his daughter, granddaughter, and so on. In particular, suppose that their well-beings induce the grandmother, who cares directly about them, to prefer c to c'. Then, the axiom says that if the son were to unfortunately die, his mother should continue to prefer $_2c$ to $_2c'$ for her granddaughter and so on.

These axioms lead to the representation in Theorem 3 below, one of the paper's main results. Intuitively, by the theorem it is as if generation 0 derives a utility u from its consumption, as usual, and an altruism utility G from the well-being of future generations, which it discounts exponentially as generations move farther away in the lineage.[18]

[18]Rogers (1994) suggests an evolutionary justification based on genetic relationship for why generation

Moreover, G is bounded, so the well-being of future generations can have only a limited impact on generation 0's well-being. Intuitively, this generation cannot become infinitely happy or unhappy just from its altruism towards its descendants. No bound applies, however, to the consumption utility u.

Theorem 3 (Additive Pure-Altruism Representation). *Axioms 1-8 hold if and only if the function U may be chosen so that*

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_t c)) \tag{5}$$

where $\alpha \in (0,1)$, $u : X \to \mathbb{R}$ and $G : \mathcal{U} \to \mathbb{R}$ are continuous, nonconstant functions, and G is strictly increasing and bounded. Moreover, if $\hat{u}$, $\hat{\alpha}$, and $\hat{G}$ represent the same $\succ$ as in (5), then $\hat{\alpha} = \alpha$ and there exist $a > 0$ and $b \in \mathbb{R}$ such that $\hat{u}(x) = au(x) + b$ and $\hat{G}(\hat{U}) = aG(\frac{\hat{U}-b}{a})$ for all c, x, and $\hat{U}$.

The theorem's proof relies on known results in Debreu (1960) and Koopmans (1960). However, a complication arises in our setting. In general, the set $\mathcal{F}$ of streams of future generations' well-being induced by consumption streams in C (see (3)) is not a Cartesian product, as generation t's well-being depends on the well-being of its descendants. Roughly speaking, to deal with this issue, the key is to show that (i) if we take any stream f in $\mathcal{F}$, there is an open neighborhood of f which belongs to $\mathcal{F}$ and has the structure of a Cartesian product, and (ii) it is possible to "cover" $\mathcal{F}$ with countably many of such neighborhoods which intersect with each other. Given (i) and (ii), we can obtain a preliminary additive representation on each neighborhood. Relying on these representations' uniqueness up to positive affine transformations, we can then "glue" all of them into a single representation over the entire set $\mathcal{F}$.[19]

One might wonder whether expression (5) is always well defined for any function G. By Theorem 1 and Axiom 4, U is a nonconstant representation of $\succ$ with values in the interval $\mathcal{U} \subset \mathbb{R}$ since X is connected. Therefore, there always exist streams c such that $U(c)$ is bounded. This implies joint restrictions on α and G. Proposition 3 identifies a sufficient (and almost necessary) restriction for (5) to be well defined. It also shows that the function U in (5) is such that the effect on current well-being of changes in the consumption of future generations becomes arbitrarily small, if such changes occur sufficiently far in the future.

0 may progressively care less about its children, grandchildren, and so on: every step in the lineage reduces the share of genes that generation 0 can expect to have in common with that generation.

[19]This approach may be useful more generally to obtain separable models of a decision-maker who exhibits pure altruism towards other individuals in society and takes into account that they are also purely altruistic towards each other.

Definition 4. A function $U : C \to \mathbb{R}$ is H-continuous if, for every $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that the following holds: if $c, \tilde{c} \in C$ satisfy $c_t = \tilde{c}_t$ for $t \leq T(\varepsilon)$, then $|U(c) - U(\tilde{c})| < \varepsilon$.[20]

Proposition 3.
(i) In representation (5), U is H-continuous and for $\nu', \nu \in \mathcal{U}$

$$|G(\nu') - G(\nu)| < \frac{1-\alpha}{\alpha} |\nu' - \nu| .$$

(ii) Suppose G is strictly increasing, bounded, and K-Lipschitz continuous with $K < \frac{1-\alpha}{\alpha}$, i.e., for all $\nu', \nu \in \mathcal{U}$

$$|G(\nu') - G(\nu)| \leq K |\nu' - \nu| .$$

Then, there exists a unique H-continuous function $U : C \to \mathbb{R}$ that solves (5).

This result helps to choose G appropriately in applications. Moreover, it has several implications regarding the properties of generation 0's preference which we present shortly.

Koopmans' Stationarity and Imperfect Altruism

Koopmans' (1960) model of intertemporal preferences is based on the general representation $U(c) = \hat{V}(u(c_0), U({}_1c))$ with $\hat{V}$ strictly increasing in each argument. To obtain such a representation in the present paper's framework, it is enough to add Koopmans' stationarity axiom to Axioms 5-6.[21]

Axiom 9 (Koopmans' Stationarity)**.** ${}_1c \succsim {}_1c'$ *if and only if* $(c_0, {}_1c) \succsim (c_0, {}_1c')$.

Of course, as Koopmans (1960) showed, if we want to specialize $\hat{V}$ to obtain the EDU model, we need stronger separability assumptions. It is easy to see why Axiom 9 rules out pure altruism. It says that, given her consumption c_0, how a grandmother ranks the consequences of streams ${}_1c$ and ${}_1c'$ for her son, granddaughter, great granddaughter, and so on is always pinned down by how she thinks that her son ranks such consequences for himself and his descendants. Therefore, the grandmother can care about the well-being of her granddaughter, great granddaughter, and so on only through how they affect her son's well-being! This property seems too strong and rather unconvincing, or at least at odds with the typical figure of a grandparent.

It is also worth emphasizing that, despite its formal similarity with time consistency, Axiom 9 is conceptually very different. Indeed, Koopmans writes,

[20] This notion is similar to that of "continuity at infinity" of payoff functions in infinite-horizon games (see, e.g., Fudenberg and Tirole (1991)).

[21] This set of axioms imply postulates 2-4 in Koopmans (1960).

> "[Stationarity] does *not* imply that, after one period has elapsed, the ordering then applicable to the 'then' future will be the same as that now applicable to the 'present' future. All postulates are concerned with only one ordering, namely that guiding decisions to be taken in the present. Any question of change or consistency of preferences as the time of choice changes is therefore extraneous to the present study." (Koopmans et al. (1964), p. 85, emphasis in the original)

It is straightforward to construct examples of preferences $\{\succ^t\}_{t=0}^{\infty}$ that satisfy Axiom 9 but are not time consistent, and vice versa.

4 Properties of the Additive Pure-Altruism Representation

This section focuses on the additive pure-altruism representation of Theorem 3. We show that it implies a number of additional properties of generation 0's preference and it can be easily applied to study intertemporal choice problems. We also show how a special case of representation (5) corresponds to the β-δ model, which was first proposed by Phelps and Pollak (1968) to capture "imperfect" intergenerational altruism and to study its effects on savings and economic growth.

4.1 Selfishness Always Dominates

We may wonder whether a directly purely-altruistic generation may be willing to sacrifice its consumption for the good of future generations, given the utility that it derives from their improved well-being. Proposition 3 implies that the answer to this question is no: a directly purely-altruistic generation 0 is always selfish, even though each future generation's well-being depends on all its descendants' well-being and generation 0 fully anticipates this.

Definition 5 (Selfishness)**.** Let $x,\ y \in X$ be such that $(x, c) \succ (y, c)$ for all $c \in C$. Then $\succ$ exhibit selfishness if, for any $t > 0$, $c^x \succ c^y$ where $c^x_0 = x$, $c^x_t = y$, $c^y_0 = y$, $c^y_t = x$, and $c^x_s = c^y_s$ otherwise.

Selfishness differs from present bias (Definition 3). It refers to a trade-off that generation 0 may face between achieving higher satisfaction itself rather than granting more satisfaction to some future generation. Present bias, by contrast, refers to how generation 0's taste for earlier vs. later satisfaction changes between a situation in which it is directly involved and a situation in which only future generations are involved.

Corollary 1. *If axioms 1-8 hold, then $\succ$ exhibits selfishness.*

Representation (5) satisfies further properties regarding how generation 0 views future generations as selfish and present biased. In a nutshell, generation 0 thinks that all generations before t agree with generation $t-1$ regarding the trade-offs that generation t faces. First, suppose that generation 0 is indifferent between c and c' which involve, for some generation $t > 0$, the same trade-off as in Definition 3. Then, generation 0 also thinks that all generations s preceding generation t will be indifferent between ${}_sc$ and ${}_sc'$ as well. Second, generation 0 views each future generation t as having its same preference and hence being selfish as stated in Definition 5. But does generation 0 expect generations $s < t$ to support generation t's selfishness? Because of time inconsistency, the answer can go either way. However, if generation 0 thinks that generation $t-1$ does (not) agree with generation t, then it also thinks that all preceding generations do (not) agree as well.[22]

4.2 A Bellman-type Equation for Optimal Consumption Policies

To see how to work with representation (5), consider the following intergenerational cake-eating problem—it should be clear that the method described here can be generalized to other Markovian decision problems. Generation 0 must commit to a policy specifying how society will consume a finite amount of resources b, the cake size. Such a policy corresponds to a stream $(c_0, c_1, \ldots) \in \mathbb{R}_+^{\mathbb{N}}$ that satisfies $\sum_{t\geq 0} c_t \leq b$. Let $C(b)$ be the set of all nonnegative consumption streams satisfying this constraint. Based on representation (5), generation 0's optimal utility is given by

$$U^*(b) = \sup_{c_0 \leq b} \{u(c_0) + \alpha A(b - c_0)\}, \tag{6}$$

where

$$A(b') = \sup_{c' \in C(b')} \sum_{t=0}^{\infty} \alpha^t G(U({}_tc')).$$

If we can solve for A, we can then easily determine the optimal allocation policy. Note

[22] In a finite-horizon setting, a representation of the form in (5) can imply that generation 0 is not selfish in the following sense: It may be willing to sacrifice its consumption for the benefit of future generations that are sufficiently close to it in the lineage. In short, this is because, with finite horizon, expression (5) can represent well-defined preferences even if the function G is "steeper" than $\frac{1-\alpha}{\alpha}$. In this case, altruism can be strong enough to overcome discounting in the near future.

that, for any $b \geq 0$, we can express $A(b)$ as

$$A(b) = \sup_{c_0 \leq b} \left\{ \sup_{c' \in C(b-c_0)} \left\{ G\left(u(c_0) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U({}_tc')) \right) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U({}_tc')) \right\} \right\}.$$

With increasing G, this yields the following Bellman-type equation for A:

$$A(b) = \sup_{c_0 \leq b} \{G(u(c_0) + \alpha A(b - c_0)) + \alpha A(b - c_0)\} \tag{7}$$

Given A, the maximization in (6) determines the optimal c_0 and that in (7) determines c_t for all $t > 0$.

Equation (7) differs from usual Bellman equations mainly because the consumption-utility term is inside the function G. Indeed, it reduces to a standard equation if G is linear. However, under minor regularity conditions on G, (7) has a well-defined solution A. To see this, define the operator $\mathcal{J}$ on the set $B(\mathbb{R}_+)$ of bounded real-valued functions of $\mathbb{R}_+$ by

$$\mathcal{J}(A)(b) = \sup_{c_0 \leq b} \{G(u(c_0) + \alpha A(b - c_0)) + \alpha A(b - c_0)\}.$$

Then, if G is bounded and K-Lipschitz continuous with $K < (1 - \alpha)/\alpha$, it is easy to show that $\mathcal{J}$ is a contraction and therefore has a unique fixed point. So equation (7) uniquely defines A. It is straightforward to approximate numerically this fixed-point, which is just a univariate function, and the rate of convergence of numerical schemes is given as a function of the Lipschitz constant of G.

When generation 0 cannot commit to a policy, time inconsistency leads to an equilibrium problem, in which each generation t chooses its c_t. Existence and properties of Markovian equilibria in a similar setting—the 'buffer-stock model,' which includes stochastic shocks to the state (b here)—have been studied by Ray (1987), Bernheim and Ray (1989), Harris and Laibson (2001), and Quah and Strulovici (2013). Bernheim and Ray study a set of utility functions that includes those in Theorem 3, so their equilibrium analysis applies to the preferences studied here.

4.3 Intergenerational Rate of Utility Substitution

Representation (5) has interesting implications on how generation 0 trades off the consumption of different future generations. By Axiom 6, separability holds between its consumption and future generations' well-being, as well as across their well-being. Nonetheless, how generation 0 trades off its consumption and that of generation t can depend on the well-being—and hence consumption—of all other generations. This is because generation t's well-being affects the well-being of all preceding altruistic generations and takes account of the well-being of all subsequent generations.

To examine this, we consider generation 0's discount function between 0 and t. Of course, intergenerational trade-offs involving consumption also depend on the curvature of u. To bypass this dependence, first note that by Theorem 3, given α and G, preference $\succ$ is entirely driven by the consumption utility u.

Corollary 2 (u-Representation)**.** *Given representation* (5)*, there exists a nonconstant function* $\hat{U} : \mathcal{I}_u^{\mathbb{N}} \to \mathbb{R}$ *(where* $\mathcal{I}_u$ *is* u*'s range) such that, for all* $c \in C$*,*

$$U(c) = \hat{U}(u(c_0), u(c_1), \ldots).$$

Relying on this result, given stream c, define $u_s = u(c_s)$ and the discount function as

$$d(t, c) = \frac{\partial \hat{U}(u_0, u_1, \ldots)/\partial u_t}{\partial \hat{U}(u_0, u_1, \ldots)/\partial u_0}. \tag{8}$$

That is, $d(t, c)$ is the marginal rate at which generation 0 substitutes consumption utility between 0 and t. Note that in the EDU model $d(t, c) = \delta^t$. For $d(t, c)$ to be well defined, the derivatives in (8) must exist. This is always the case when G is differentiable.[23]

Proposition 4. *Suppose G in representation (5) is differentiable. Then,* $d(1, c) = \alpha G'(U({}_1c))$ *and, for* $t > 1$*,*

$$d(t, c) = \alpha^t G'(U({}_tc)) \left[1 + \sum_{\tau=1}^{t-1} G'(U({}_{t-\tau}c)) \prod_{s=1}^{\tau-1} (1 + G'(U({}_{t-s}c)))\right], \tag{9}$$

where $\prod_{s=1}^{\tau-1}(1 + G'(U({}_{t-s}c))) \equiv 1$ if $\tau = 1$.

This formula has a simple explanation. Suppose $u(c_t)$ rises by a small amount. This has two effects from generation 0's viewpoint: (1) generation t's well-being rises, which explains the term $G'(U({}_tc))$; consequently, (2) well-being also rises for each generation τ between 1 and t, which explains the summation in (9). Moreover, the rise in $U({}_tc)$ affects $U({}_{t-\tau}c)$ through all well-beings of generations between $t - \tau$ and t, which explains the product in (9).

By Proposition 4, for general G the discount function $d(t, c)$ depends not only on generation t's consumption, but also on all other future generations' well-being—hence it depends on the entire stream c. How $d(t, c)$ varies with c ultimately depends on the properties of G. If G' is decreasing (increasing), then generation 0 discounts more a stream yielding higher (lower) consumption utility to all generations. To state this formally, for any c, $c' \in C$, let $c \geq_u c'$ if and only if $u(c_t) \geq u(c'_t)$ for all $t \geq 0$.

[23]Note that, being increasing, G is already differentiable at almost every point in $\mathcal{U}$.

Corollary 3. *Let $d(t,c)$ be as in Proposition 4. For any $t>0$, $c \geq_u c'$ implies $d(t,c) \leq d(t,c')$ if and only if G' is decreasing. Conversely, $c \geq_u c'$ implies $d(t,c) \geq d(t,c')$ if and only if G' is increasing.*

This result shows a tight link between discounting and G's curvature, which may be empirically tested. For instance, a concave G implies that, after learning that the living standards of future generations will not improve as predicted by the historical trend, generation 0 may become *more* willing sacrifice its current satisfaction (i.e., $u(c_0)$) to improve that of future generations (i.e., $u(c_t)$ for $t>0$).

The only case in which $d(t,c)$ is independent of c is when G is linear. Surprisingly, in this case, the discount factor takes a very well-known form.

Corollary 4. *Suppose $G(U)=\gamma U$ with $\gamma \in (0, \frac{1-\alpha}{\alpha})$. Then, for all $t>0$,*

$$d(t,c)=\beta\delta^t,$$

where $\beta=\frac{\gamma}{1+\gamma}$ and $\delta=(1+\gamma)\alpha<1$.

Proof. By Proposition 4, the result is immediate for $t=1,2$. For $t>2$,

$$d(t,c)=\alpha^t\gamma\left[1+\gamma\sum_{\tau=1}^{t-1}(1+\gamma)^{\tau-1}\right]=\alpha^t\gamma(1+\gamma)^{t-1}.$$

□

Thus the class of preferences that Phelps and Pollak (1968) invented to model "imperfectly" altruistic generations is actually equivalent to a specific version of preferences that exhibit direct pure altruism. Perhaps ironically, by contrast, the EDU model, which Phelps and Pollak (1968) viewed as modeling "perfectly" altruistic generations, is actually a specific version of preferences that exhibit only indirect pure altruism. Thus the present paper completely reverses the common view on which between exponential and β-δ discounting captures a stronger degree of intergenerational altruism.

4.4 Quasi-hyperbolic Discounting of Consumption Utilities

Corollary 4 raises a natural question: Which properties of $\succ$ correspond to linearity of G and hence to quasi-hyperbolic discounting? As noted, unless G is linear, how generation 0 trades off its consumption and that of generation t depends on the well-being of all other generations. This observation suggests Axiom 10.

Axiom 10 (Consumption Independence).
(i) $(c_0, c_1, {}_2c) \succ (c'_0, c'_1, {}_2c)$ *if and only if* $(c_0, c_1, {}_2c') \succ (c'_0, c'_1, {}_2c')$;
(ii) $(c_0, c_1, {}_2c) \succ (c'_0, c_1, {}_2c')$ *if and only if* $(c_0, c'_1, {}_2c) \succ (c'_0, c'_1, {}_2c')$.

Intuitively, condition (i) says a grandmother trades off her consumption with that of her son in a way that does not depend on the consumption and hence well-being of her son's descendants. Condition (ii) says that she trades off her consumption with that of her son's descendants in a way that does not depend on her son's consumption.

Theorem 4 (Linear Pure-Altruism Representation). *Axiom 1-8 and 10 hold if and only if the function* U *may be chosen so that*

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t \gamma U({}_tc), \tag{10}$$

where $\alpha \in (0,1)$, $\gamma \in (0, \frac{1-\alpha}{\alpha})$, *and* $u : X \to \mathbb{R}$ *is a continuous nonconstant function.*

Corollary 5 (Quasi-hyperbolic Discounting). *Axiom 1-8 and 10 hold if and only if there are* β, $\delta \in (0,1)$ *and a continuous nonconstant function* $u : X \to \mathbb{R}$ *such that*

$$U(c) = u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t).$$

Proof. By Theorem 4, for all t, $U(c)$ is a strictly increasing, linear function of $U({}_tc)$, which is in turn a strictly increasing, linear function of $u(c_t)$. Hence, there exists a function $\kappa(t) : \mathbb{N} \setminus \{0\} \to \mathbb{R}_{++}$ such that

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \kappa(t) u(c_t).$$

Clearly, for all $t > 0$, $\kappa(t) = d(t, c)$ defined in (8). Corollary 4 implies the result. □

This result allows us to understand β-δ discounting of each generation's consumption utility in terms of simple properties of generation 0's directly purely-altruistic preference. This preference depends on two, conceptually different, entities: generation 0's physical consumption and the overall well-being of all future generations, about which generation 0 cares due to altruism. By rationally expecting all future generations to be altruistic as well, however, generation 0 develops the view that future generations should be more willing to sacrifice their consumption for the good of their descendant than how much it itself is willing to sacrifice (present bias). Thus generation 0 treats its consumption utility and that of all future generations in a systematically different way, a way that is, however, coherent across generations as captured by our altruism stationarity (Axiom 8). This explains what Phelps and Pollak (1968) called "imperfect" altruism ($\beta < 1$) and the stark form it takes in their model. Moreover, generation 0 evaluates its consumption and the well-being of future generations in a separable way, and trades off its consumption against that of any future generation in a way that does not depend on other generations' well-being. This delivers the additive separability in terms of consumption utilities.

Corollary 5 provides a tight link between generation 0's degree of present bias, β, and its marginal utility from altruism, γ. In this linear representation, it is possible to interpret γ as the degree to which generation 0 finds future generations' well-being "imaginable" or "vivid." Note that this is distinct from how much generation 0 cares about its descendants as the generational distance increases, which is captured by α^t. Thus, we identify two possible determinants of how much generation 0 is altruistic towards generation t. Corollary 5 implies that β is directly proportional to γ, so generation 0 becomes less present biased when it anticipates more easily the well-being of future generations.[24] For its preference to be well-defined, however, γ cannot rise above a certain level unless generation 0 discounts at a faster rate α its distance from future generations.

Comparison with Other Axiomatizations of β-δ Discounting

Other papers axiomatize intertemporal preferences that correspond to the β-δ model (for example, see Hayashi (2003); Olea and Strzalecki (2014)). These papers, however, take the perspective of a single individual, not of different generations. To attempt a comparison with the present paper, note that generation 0—the decision-maker in our framework—essentially treats its descendants as future copies of itself. Thus, from a single-individual perspective, one may view our analysis as describing an agent who perceives himself as a collection of selves, one for each period t, and exhibits direct pure altruism towards his future selves, expecting that they will do the same. Therefore, self 0 directly cares about the well-being (i.e., $U({}_tc)$) of all his future selves through *intra*personal altruism,[25] an attitude that is revealed by his current, observable preference over consumption streams.

This property that self 0 directly cares about all his future selves' well-being would be the first, key conceptual difference from the previous axiomatizations of β-δ discounting.

[24] Vividness of the well-being of future generations implied by today's decisions may be influenced with specific information campaigns. For example, consider the dramatic images that media and environmental organizations report on the catastrophic consequences of manmade climate change.

[25] The idea that an individual over time consists of a collection of selves is not new in economics (see, for example, Strotz (1955) and Frederick (2003)) and has a long tradition in philosophy (see, for example, Parfit (1971, 1976, 1982)). In this case, direct pure altruism simply means that the individual's present self cares about his future selves and fully anticipates that they will continue to do the same.

Both Hayashi (2003) and Olea and Strzalecki (2014) adopt the common view that the decision-maker cares about his per-period consumption utilities.[26] Within this framework, they replace Koopmans' (1960) stationarity (Axiom 9) with quasi-stationarity, namely stationarity from period 1 onward. However, quasi-stationarity seems difficult to justify when the decision-maker evaluates streams based only on consumption utilities. If he views consumption in the same way in all periods, why should stationarity hold between period 1 and later periods, but not between period 0 and 1? This issue does not arise with altruism stationarity (Axiom 8), for self 1's well-being is equivalent to the well-being of all future selves, but is different from self 0's immediate consumption.

Second, to obtain the β-δ representation, Olea and Strzalecki (2014) need to ensure that current and future per-period utilities are cardinally equivalent. Their ingenious axioms permit useful experiments to identify and measure β and δ, but seem difficult to interpret. Moreover, they introduce an explicit present-bias axiom to obtain $\beta \leq 1$. By contrast, when self-0 directly cares about the well-being he ascribes to all his future, similarly altruistic selves, present bias follows as a general, logical consequence (Proposition 2). Moreover, β-δ discounting is tightly linked with an intuitive consumption-independence condition (Axiom 10).

Echenique et al. (2014) propose a different, interesting method to characterize several models of intertemporal choice, including the EDU and β-δ model. Their starting point is a data set consisting of a decision-maker's choices of finite consumption streams from standard budget sets and the prices of consumption at all dates that define such sets. They then identify versions of the Generalized Axiom of Revealed Preference that the data must satisfy to be consistent with the EDU and β-δ model, respectively. Using real data from the experiment in Andreoni and Sprenger (2012), they apply their axioms to classify subjects across models: roughly only one third of the 97 subjects is consistent with either EDU or β-δ discounting, and about a half violates time separability in consumption.

[26] This seems, at least, the most natural interpretation of the second paper, which is based on the idea of annuity compensation: to avoid relying on assumptions of the form of u to elicit β and δ separately, the idea is to consider fix compensation levels—and hence fixed u's—and vary the time horizon at which they occur, so as to find exact points of indifference for the decision-maker and hence infer the parameters of the model. The decision-maker has different subjective views of the time distance between period 0 and 1 and between any two future periods. If he cares only about the u's that he gets in each period, then it is possible to objectively space out these u's in an appropriate way so as to identify β and δ.

5 Welfare Analysis with Intergenerational Altruism

Models that allow for time-inconsistent preferences pose serious conceptual problems when defining welfare criteria and addressing policy questions (see, e.g., Rubinstein (2003); Bernheim and Rangel (2007, 2009)). Discussing β-δ discounting, Rubinstein (2003) notes,

> "Policy questions were freely discussed in these models even though welfare assessment is particularly tricky in the presence of time inconsistency. The literature often assumed, though with some hesitation, that the welfare criterion is the utility function with stationary discounting rate δ (which is independent of β)." (p. 1208)

Other, perhaps more fundamental, issues are whether time inconsistency across generations' preferences justifies some form of paternalistic interventions by a social planner, and whether such preferences may have other properties that are normatively appealing. Identifying and assessing rigorously these properties may be useful, for instance, if the planner responds only to the preference of the currently alive generation, as in the case of democratic governments.

An immediate consequence of the present paper is to *weaken* the case for paternalistic interventions. If intergenerational time inconsistency were the result of some form of bounded rationality or lack of consideration for future generations, one might be tempted to argue that society can benefit from delegating its choices to a paternalistic planner. This argument, however, is invalid if time inconsistency is the logical consequence of direct pure altruism which every generation fully takes into account. In this case, the currently alive generation 0 is already taking account of the preferences that it expects all future generations to have, a property that is revealed through its current preference. Thus, unless the planner has good reasons to believe that generation 0 anticipates future generations' preferences in a systematically incorrect way, we may argue that she should simply adopt a "libertarian" stance and measure the welfare of any stream c using $W^L(c) = U(c)$. This is the welfare criterion that is usually and uncontroversially applied for time-consistent models, such as EDU.

Given the relationship between time consistency and the degree of intergenerational altruism highlighted in this paper, a libertarian criterion W^L seems even more appropriate for our model than for the standard, time-consistent model. For the latter model W^L corresponds to evaluating consumption streams for the entire society based only on generation 0's consumption utility and how it thinks that generation 1 will rank the remaining entire streams. For our model, by contrast, W^L evaluates streams based on generation 0's consumption utility and how it thinks that *all* future generations will rank the streams they will face for their remaining future. For example, consider two policies, A and B, inducing streams c^A and c^B such that ${}_1c^A \sim {}_1c^B$ but ${}_tc^A \succ {}_tc^B$ for all $t > 1$. Then the W^L based on the standard, time-consistent model (such as EDU) implies that policy A is as desirable as B. By contrast, the W^L based on our model implies that A is strictly more desirable. Thus, the second criterion is more sensitive to the long-run consequences of policies and treats future generations in a more fair way rather than catering entirely to generation 1. These properties seem appealing for a welfare criterion.

In the case of our additive pure-altruism representation in Theorem 3, its core axioms also suggest that generation 0 treats future generations in an equitable and coherent way, thereby adding normative appeal to the criterion $W^L(c) = U(c)$ for this representation. Intergenerational separability (Axiom 6) implies that the well-being of some generation t affect neither how generation 0 evaluates its own physical consumption nor how it cares about other generations' well-being. Altruism stationarity (Axiom 8) implies that generation 0 treats the well-being of all future generations in a coherent way.

Despite generation 0's direct pure altruism, one may still argue that a libertarian criterion does not sufficiently take into account the preferences of future generations. Thus, we may want to consider paternalistic welfare criteria that aggregate the preferences (i.e., well-being) across all generations. A natural question is also whether we can find an aggregator which, despite the preferences' time inconsistency, renders the social planner time *consistent*. A simple candidate seems

$$W^D(c) = \sum_{t=0}^{\infty} w(t) U({}_tc), \tag{11}$$

where $w : \mathbb{N} \to \mathbb{R}_{++}$. In this case, we obtain the following.

Proposition 5. *Suppose $U(c)$ can be represented as in Theorem 4 and Corollary 5, with corresponding parameters (u, α, γ) and (u, β, δ). Then,*

$$W^D(c) = \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{12}$$

if and only if $w(t) = \alpha^t$.

One direction of this result follows from an intuitive argument—the other is provided in Appendix A. Let $W(c) = \sum_{t=0}^{\infty} \alpha^t U({}_tc)$. Using α-γ representation of $U(c)$ in Theorem 4, we have

$$W(c) = u(c_0) + (1+\gamma)\alpha \sum_{t\geq 1} \alpha^{t-1} U({}_tc) = u(c_0) + (1+\gamma)\alpha W({}_1c).$$

This shows why the planner is time consistent and $W(c)$ corresponds to the sum of consumption utilities exponentially discounted with factor $\hat{\delta} = (1+\gamma)\alpha$. Of course, we know from Corollaries 4 and 5 that $\hat{\delta} = \delta$ in the β-δ version of $U(c)$. However, to see why this has to be the case for any $\hat{\delta}$, note that the α-γ representation satisfies

$$U(c) = u(c_0) + \gamma\alpha W({}_1c) = u(c_0) + \frac{\gamma\alpha}{\hat{\delta}} \sum_{t>0} \hat{\delta}^t u(c_t).$$

It follows that $U(c)$ must have a quasi-hyperbolic representation in terms of consumption utilities, where the long-run discount factor coincides with the planner's factor.

Proposition 5 is remarkable for several reasons. First, for a specific, though renowned, model of intergenerational altruism, there exists a paternalistic welfare criterion that aggregates the time-inconsistent preferences of all generations, yet renders the social planner time consistent. Moreover, this criterion is completely determined by properties of generation 0's revealed preference: its consumption utility, u, its constant marginal utility of altruism, γ, and the rate at which it discounts the gap from future generations, α. Second, it is well-known that, if we want the social planner to be time consistent, to treat the consumption of different generations in an additively separable way, and to conform to each generation's consumption utility, then W^D must take the form of exponential discounting as in (12) for some discount factor $\hat{\delta}$. This renders the criterion in (12) a natural and appealing benchmark.

Third, a result similar to Proposition 5 does not hold for more general representations of directly purely-altruistic preferences, such as that in Theorem 3. Even if we consider $W^D(c) = \sum_{t=0}^{\infty} \alpha^t U({}_tc)$,[27] when $U({}_tc)$ can be represented as in (5) with a general function G, the planner need not satisfy the standard notion of time consistency—that is, $W^D(c_0, {}_1c) \geq W^D(c_0, {}_1c')$ if and only if $W^D({}_1c) \geq W^D({}_1c')$. The reason is that generation 0 and the planner trade off the well-being of future generations in different ways. Thus, when the planner has to take into account the preference of generation 0, she may strictly prefer $(c_0, {}_1c)$ to $(c_0, {}_1c')$. However, if the planner disregards generation 0's opinion, she may strictly prefer ${}_1c'$ to ${}_1c$.[28]

One may wonder whether in the case of indirect pure altruism it is possible to aggregate the preferences of all time-consistent generations with a criterion that renders the planner time consistent. In the case of EDU—i.e., $U({}_tc) = \sum_{t=0}^{\infty} \delta^t u(c_t)$—one might rely on δ to aggregate well-being across generations using the criterion $\hat{W}(c) = \sum_{t=0}^{\infty} \delta^t U({}_tc)$ (see,

[27]Note that a welfare criterion of the form $W(c) = \sum_{t=0}^{\infty} r^t U({}_tc)$ with $0 < r < 1$ trades off streams of *well-being* in a time-consistent manner.

[28]Formally, it is possible to construct examples in which $\sum_{t=1}^{\infty} \alpha^t G(U({}_tc)) > \sum_{t=1}^{\infty} \alpha^t G(U({}_tc'))$ and $\sum_{t=1}^{\infty} \alpha^t [G(U({}_tc)) + U({}_tc)] > \sum_{t=1}^{\infty} \alpha^t [G(U({}_tc')) + U({}_tc')]$, but $\sum_{t=1}^{\infty} \alpha^t U({}_tc) < \sum_{t=1}^{\infty} \alpha^t U({}_tc')$. In this case, $W^D(c_0, {}_1c) > W^D(c_0, {}_1c')$, but $W^D({}_1c) < W^D({}_1c')$.

e.g., Ray (2014)). Doing so, however, makes the planner time inconsistent. Indeed, one can show that $\hat{W}(c) = u(c_0) + \sum_{t=1}^{\infty} \delta^t (1+t) u(c_t)$.

In short, except for the case of Phelps and Pollack's (1968) quasi-hyperbolic model, pure altruism across generations—whether direct or indirect—raises serious challenges if we want to construct a welfare criterion that both takes into account the preferences of all generations and renders a planner time consistent.

A Appendix: Proofs of the Main Results

A.1 Proof of Proposition 2

Let $U(c) = V(c_0, U({}_1c), U({}_2c), \ldots)$ where V is strictly increasing in $U({}_tc)$ for all $t > 0$. By definition, $(x,c) \succ (y,c)$ means that $U(x,c) > U(y,c)$. Hence, for all $0 \le s \le t$,

$$U({}_sz_t, x, c) > U({}_sz_t, y, c),$$

where, for $s < t$, ${}_sz_t = (z_s, \ldots, z_t)$ and ${}_tz_t = z_t$. This follows by induction. For $s = t$,

$$U({}_tz_t, x, c) = V({}_tz_t, U(x,c), U(c), \ldots) > V({}_tz_t, U(y,c), U(c), \ldots) = U({}_tz_t, y, c).$$

Now suppose that the claim holds for $r+1 \le s \le t$, with $0 \le r < t$. Then

$$\begin{aligned} U({}_rz_t, x, c) &= V(z_r, U({}_{r+1}z_t, x, c), \ldots, U({}_tz_t, x, c), U(x,c), \ldots) \\ &> V(z_r, U({}_{r+1}z_t, y, c), \ldots, U({}_tz_t, y, c), U(y,c), \ldots) \;=\; U({}_rz_t, y, c). \end{aligned}$$

By definition, $({}_0z_t, x, w, c') \sim ({}_0z_t, y, h, c')$ means that

$$\begin{aligned} &V(z_0, U({}_1z_t, x, w, c'), \ldots, U({}_tz_t, x, w, c'), U(x, w, c'), U(w, c'), \ldots) \\ &= V(z_0, U({}_1z_t, y, h, c'), \ldots, U({}_tz_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots). \end{aligned}$$

Since $U({}_0z_t, x, c) > U({}_0z_t, y, c)$ for all c,

$$\begin{aligned} &V(z_0, U({}_1z_t, y, w, c'), \ldots, U({}_tz_t, y, w, c'), U(y, w, c'), U(w, c'), \ldots) \\ &< V(z_0, U({}_1z_t, y, h, c'), \ldots, U(z_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots). \end{aligned}$$

This implies that $U(h, c') > U(w, c')$. Otherwise, $U(y, h, c') \le U(y, w, c')$ and, by induction, $U({}_sz_t, y, h, c') \le U({}_sz_t, y, w, c')$ for all $0 \le s \le t$, which is a contradiction.

Finally, we must have $U(x, w, c') > U(y, h, c')$. Otherwise, again by induction, for all $0 \le s \le t$

$$U({}_sz_t, y, h, c') > U({}_sz_t, x, w, c'),$$

which contradicts $({}_0z_t, x, w, c') \sim ({}_0z_t, y, h, c')$.

Suppose that we replace condition $({}_0z_t, x, w, c') \sim ({}_0z_t, y, h, c')$ with $({}_0z_t, x, {}_{t+2}z_s, w, c') \sim ({}_0z_t, y, {}_{t+2}z_s, h, c')$ where $s \ge t+2$. By the same argument as before, $({}_0z_t, y, {}_{t+2}z_s, w, c') \prec ({}_0z_t, y, {}_{t+2}z_s, h, c')$ and so $(h, c') \succ (w, c')$. If not, by induction $({}_\tau z_s, w, c') \succsim ({}_\tau z_s, h, c')$ for all $0 \le \tau \le s$ (where $z_{t+1} = y$). Then, we must have $(x, {}_{t+2}z_s, w, c') \succ (y, {}_{t+2}z_s, h, c')$. If not, since $({}_\tau z_s, h, c') \succ ({}_\tau z_s, w, c')$ for all $t+2 \le \tau \le s$, we would have $({}_0z_t, y, {}_{t+2}z_s, h, c') \succ$

$({}_0z_t, x, {}_{t+2}z_s, w, c')$.

A.2 Proof of Theorem 3

By the definition of $\mathcal{F}$ in (3), note that $\mathcal{F}$ need not be a Cartesian product, as f_t depends on f_s for $s > t$. Letting ${}_tf = (f_t, f_{t+1}, \ldots)$, we can denote elements in $\mathcal{F}$ by $(f_1, f_2, \ldots, f_{t-1}, {}_tf)$. On $\mathcal{F}_0 = X \times \mathcal{F}$ (where $f_0(c) = c_0$), the primitive $\succ$ induces a $\succ^*$ with representation $V : \mathcal{F}_0 \to \mathbb{R}$; by Theorem 2, $\succ^*$ is well defined. If $\mathcal{F}_0$ were a Cartesian product, we could mimic the steps in Debreu (1960) (Theorem 3) and Koopmans (1972) on the domain $\mathcal{F}_0$ to prove our theorem. However, this is not possible. We will then proceed as follows. In step 1, we show that $\succ^*$ satisfies the essentiality and strong separability properties at the heart of Debreu's (1960) Theorem 3. In step 2, we show that the ranking of streams $(f_0, f_1, f_2, {}_3f) \in \mathcal{F}_0$ depends only on a function of ${}_3f$; so we can restrict attention to a four dimensional space. In step 3, we show that this space is a Cartesian product 'locally;' so we can apply Debreu's result to obtain an additive representation 'locally.' Since additive representations are unique up to positive, affine transformations, we can extend uniquely the additive representation to the entire $\mathcal{F}_0$. In step 4, we show that this representation takes the form in our Theorem 3.

Step 1. Lemma 1 says that, if $(f_0, f_1, f_2, {}_3f) \succ^* (f'_0, f'_1, f'_2, {}_3f')$, then changing the common components of $(f_0, f_1, f_2, {}_3f)$ and $(f'_0, f'_1, f'_2, {}_3f')$ in the same way leaves the ranking under $\succ^*$ unchanged.

Lemma 1. *Fix any nonempty subset π of $\{0, 1, 2, 3\}$. Then*

$$(f_0, f_1, f_2, {}_3f) \succ^* (f'_0, f'_1, f'_2, {}_3f') \Leftrightarrow (\hat{f}_0, \hat{f}_1, \hat{f}_2, {}_3\hat{f}) \succ^* (\hat{f}'_0, \hat{f}'_1, \hat{f}'_2, {}_3\hat{f}'),$$

where $f_t = \hat{f}_t$, $f'_t = \hat{f}'_t$, ${}_3f = {}_3\hat{f}$, and ${}_3f' = {}_3\hat{f}'$ if t or 3 are in π, and $f_t = f'_t$, $\hat{f}_t = \hat{f}'_t$, ${}_3f = {}_3f'$, and ${}_3\hat{f} = {}_3\hat{f}'$ if t or 3 are not in π.

Proof. Recall that ${}_tc \sim {}_tc'$ is equivalent to $f_t = f'_t$. Then, by Axiom 6, for any π

$$V(f_0, f_1, f_2, {}_3f) > V(f'_0, f'_1, f'_2, {}_3f') \Leftrightarrow V(\hat{f}_0, \hat{f}_1, \hat{f}_2, {}_3\hat{f}) > V(\hat{f}'_0, \hat{f}'_1, \hat{f}'_2, {}_3\hat{f}').$$

□

Using Lemma 1 with $\pi = \{0\}$ and $\pi = \{1, 2, 3\}$, we obtain the following.

Lemma 2. *The function $V : \mathcal{F}_0 \to \mathbb{R}$ can be written in the form*

$$V(f) = W(u(f_0), d({}_1f)), \tag{13}$$

where $u : X \to \mathcal{I}_u \subset \mathbb{R}$ *and* $d : \mathcal{F} \to \mathcal{D} \subset \mathbb{R}$. W *is jointly continuous in its two arguments and strictly increasing in each of them,* u *is continuous, and* $\mathcal{I}_u$ *and* $\mathcal{D}$ *are non-degenerate intervals.*

Proof. Consider $\succ^*$ on $X \times \mathcal{F}$ and Lemma 1 with $\pi = \{0\}$ and $\pi = \{1, 2, 3\}$. By an argument similar to that in Section 5 of Koopmans' (1960), for any f we can write $V(f) = W(u(f_0), d({}_1f))$, where $u(f_0) = V(f_0, {}_1\hat{f})$ for some ${}_1\hat{f} \in \mathcal{F}$ and $d({}_1f) = V(f'_0, {}_1f)$ for some $f'_0 \in X$. Recall that $V(f(c)) = U(c)$ for all $c \in C$. Hence, the continuity property of U implies continuity of u. By Axiom 4, neither u nor d can be constant. Since X is connected, u takes all values in a connected interval $\mathcal{I}_u \subset \mathbb{R}$. Since $d({}_1f(c)) = U(f'_0, {}_1c)$, U is continuous, and X is connected, d takes all values in a connected interval $\mathcal{D} \subset \mathbb{R}$. By definition of u and Lemma 1 with $\pi = \{0\}$, W must be strictly increasing in its first argument on $\mathcal{I}_u$. Similarly, by definition of d and Lemma 1 with $\pi = \{1, 2, 3\}$, W must be strictly increasing in its second argument on $\mathcal{D}$. Given $\hat{c}$, $U(\cdot, {}_1\hat{c})$ takes values in an interval. Then the strictly increasing $W(\cdot, d({}_1f(\hat{c})))$ also takes values in an interval and hence must be continuous in its first argument on $\mathcal{I}_u$. By a similar argument, W must be continuous in its second argument, and hence jointly continuous on $\mathcal{I}_u \times \mathcal{D}$.

□

Hereafter, let $\overline{u} = \sup \mathcal{I}_u$ and $\underline{u} = \inf \mathcal{I}_u$. Also note that the function d in Lemma 2 defines a ranking on $\mathcal{F}$.

Lemma 3. *There exist* $x, y, z, x', y', z' \in X$ *and* $c \in C$ *such that (i)* $(z, c) \succ (z', c)$*, (ii)* $(y, z, c) \sim (y'z', c)$*, and (iii)* $(x, y, z, c) \sim (x', y', z', c)$*.*

Proof. By Axiom 4, there exist z, $z' \in X$ and $c \in C$ such that $(z, c) \succ (z', c)$. Using representation (13), we have $u(z) > u(z')$. Now, consider (y', z, c) and (y, z', c) where $y = z$ and $y' = z'$. By Axiom 7(i), $(y', z, c) \succ (y', z', c)$.

Case 1: $(y, z', c) \succsim (y', z, c)$. Since $\mathcal{I}_u$ is connected, we can modify y to $y'' \in X$ so that $u(y'')$ takes any value in $[u(y'), u(y)]$. By Axiom 2, there exists y'' such that $(y'', z', c) \sim (y', z, c)$; moreover, we must have $u(y'') > u(y')$. Now consider (x, y'', z', c) and (x', y', z, c) where $x = z$ and $x' = z'$. By Axiom 7(i), $(x', z, c) \succ (x', z', c)$; so, by Axiom 8, $(x', y', z, c) \succ (x', y'', z', c)$.

Case 1.1: $(x, y'', z', c) \succsim (x', y', z, c)$. Since we can modify x to $x'' \in X$ so that $u(x'')$ takes any value in $[u(x'), u(x)]$, by Axiom 2, there exists x'' such that $(x'', y'', z', c) \sim (x', y', z, c)$.

Case 1.2: $(x, y'', z', c) \prec (x', y', z, c)$. We can modify z and y'' to $\tilde{y}, \tilde{z} \in X$ so that $u(\tilde{z})$ and $u(\tilde{y})$ take any value in $[u(z'), u(z)]$ and $[u(y'), u(y'')]$. Moreover, we can do so maintaining $(\tilde{y}, z', c) \sim (y', \tilde{z}, c)$ by Axiom 2. Since $(x, y', z', c) \succ (x', y', z', c)$, by Axiom 2, there exist $\tilde{y}$ and $\tilde{z}$ such that $(x, \tilde{y}, z', c) \sim (x', y', \tilde{z}, c)$. Finally, we must have $u(\tilde{z}) > u(z')$, so $(\tilde{z}, c) \succ (z', c)$.

Case 2: $(y, z', c) \prec (y', z, c)$. We can modify z to $\hat{z} \in X$ so that $u(\hat{z})$ takes any value in $[u(z'), u(z)]$. Since $(y, z', c) \succ (y', z', c)$, by Axiom 2 there exists $\hat{z}$ such that $(y, z', c) \sim (y', \hat{z}, c)$, and by Axiom 7(i) we must have $(\hat{z}, c) \succ (z', c)$ and hence $u(\hat{z}) > u(z')$. Now consider (x, y, z', c) and $(x', y', \hat{z}, c)$ where $x = z$ and $x' = z'$.

Case 2.1: $(x, y, z', c) \succsim (x', y', \hat{z}, c)$. We can modify x to $\hat{x}$ so that $u(\hat{x})$ takes any value in $[u(x'), u(x)]$. By Axiom 7(i), $(x', z', c) \prec (x', \hat{z}, c)$; so, by Axiom 8, $(x', y, z', c) \prec (x', y', \hat{z}, c)$. Then, by Axiom 2 there exists $\hat{x}$ such that $(\hat{x}, y, z', c) \sim (x', y', \hat{z}, c)$.

Case 2.2: $(x, y, z', c) \prec (x', y', \hat{z}, c)$. We can modify y and $\hat{z}$ to $\hat{y}$ and $\hat{z}'$ so that $u(\hat{y})$ and $u(\hat{z}')$ take any value in $[u(y'), u(y)]$ and $[u(z'), u(\hat{z})]$. Moreover, we can do so maintaining $(\hat{y}, z', c) \sim (y', \hat{z}', c)$ by Axiom 2. Since $(x, y', z', c) \succ (x', y', z', c)$, by Axiom 2 there exists $\hat{y}$ and $\hat{z}'$ such that $(x, \hat{y}, z', c) \sim (x', y', \hat{z}', c)$.

□

Hereafter, for $t \in \{0, 1, 2, 3\}$, we will refer to the factor t of $\mathcal{F}_0$ as the component of position $t+1$ in the representation $(f_0, f_1, f_2, {}_3f)$ of every $f \in \mathcal{F}_0$ (e.g., the factor 2 is the third component of every $(f_0, f_1, f_2, {}_3f) \in \mathcal{F}_0$).

Definition 6 (Debreu (1960))**.** For $t \in \{0, 1, 2\}$, if $f \succ^* f'$ for some $f, f' \in \mathcal{F}_0$ with $f_s = f'_s$ for all $s \neq t$, then the factor t of $\mathcal{F}_0$ is called essential for $\succ^*$. If $(f_0, f_1, f_2, {}_3f) \succ^* (f'_0, f'_1, f'_2, {}_3f')$ for some $f, f' \in \mathcal{F}_0$ with $f_s = f'_s$ for $s = 0, 1, 2$, then the factor 3 is called essential for $\succ^*$.

Lemma 4. *For all $t \in \{0, 1, 2, 3\}$, the factor t of $\mathcal{F}_0$ is essential.*

Proof. By Axiom 4, the factor 0 is essential. Using the streams in Lemma 3, let ${}_1c = (x, y, z, c)$ and ${}_1c' = (x', y', z', c)$ and consider the corresponding f and f' in $\mathcal{F}_0$ with any $f_0 = f'_0$. We have $f_1 = f'_1$, $f_2 = f'_2$, $f_3 > f'_3$, and $f_t = f'_t$ for all $t > 3$. By Axiom 7(i), $(f_0, f_3, f_4, {}_5f) \succ^* (f'_0, f'_3, f'_4, {}_5f')$, hence the factor 1 is essential. By Axiom 8, $(f_0, f_2, f_3, {}_4f) \succ^* (f'_0, f'_2, f'_3, {}_4f')$ and $(f_0, f_1, f_2, {}_3f) \succ^* (f'_0, f'_1, f'_2, {}_3f')$. So the factors 2 and 3 are essential.

□

Step 2. By Lemma 1 with $\pi = \{2, 3\}$, $\succ^*$ also satisfies the following property:

$$(f_0, f_1, {}_2f) \succ^* (f_0, f_1, {}_2f') \Leftrightarrow (\hat{f}_0, \hat{f}_1, {}_2f) \succ^* (\hat{f}_0, \hat{f}_1, {}_2f').$$

Define $\tilde{\mathcal{Q}} = \{(f_1(c), d({}_2f(c))) : c \in C\}$. Note that $\tilde{\mathcal{Q}} \subset \mathcal{U} \times \mathcal{D}$, but it need not be a Cartesian product because the value of d affects that of f_1.

Lemma 5. *There exists a continuous function $\tilde{V} : X \times \tilde{\mathcal{Q}} \to \mathbb{R}$ such that, for all $f \in \mathcal{F}_0$,*

$$V(f) = \tilde{V}(f_0, f_1, d({}_2f)), \tag{14}$$

where d is the function defined in Lemma 2. For any f_1, f_1', d', and d'' we have the following:
(5.i) if (f_1, d'') and (f_1', d'') are in $\tilde{\mathcal{Q}}$, $\tilde{V}(f_0, f_1, d'') > \tilde{V}(f_0, f_1', d'')$ iff[29] *$f_1 > f_1'$;*
(5.ii) if (f_1, d') and (f_1, d'') are in $\tilde{\mathcal{Q}}$, $\tilde{V}(f_0, f_1, d') > \tilde{V}(f_0, f_1, d'')$ iff $d' > d''$.

Proof. Recall that $d(\cdot)$ defines a ranking on $\mathcal{F}$ and that ${}_2f \in \mathcal{F}$. For any $(f_0, f_1, {}_2f)$ and $(f_0', f_1', {}_2f')$ such that both $(f_0, f_1, {}_2f')$ and $(f_0', f_1', {}_2f)$ are in $\mathcal{F}_0$, by Lemma 1 with $\pi = \{2, 3\}$, $V(f_0, f_1, {}_2f) \geq V(f_0, f_1, {}_2f')$ iff $V(f_0', f_1', {}_2f) \geq V(f_0', f_1', {}_2f')$. Moreover, for any $(f_1, {}_2f)$ and $(f_1, {}_2f')$ in $\mathcal{F}$, by Axiom 8, $V(f_0, f_1, {}_2f) \geq V(f_0, f_1, {}_2f')$ iff $W(u(f_0), d({}_2f)) \geq W(u(f_0), d({}_2f'))$, and therefore iff $d({}_2f) \geq d({}_2f')$. So, the ranking of $(f_0, f_1, {}_2f)$ and $(f_0, f_1, {}_2f')$ depends only on the value of $d(\cdot)$. Now, for any $f \in \mathcal{F}_0$, set

$$\tilde{V}(f_0, f_1, d({}_2f)) = V(f_0, f_1, {}_2f).$$

The previous argument implies property (5.i).

$\tilde{V}$ is well defined for the following reasons. First, if $(f_0, f_1, {}_2f)$ and $(f_0', f_1', {}_2f')$ are such that $f_t = f_t'$ for $t = 0, 1$ and $d({}_2f) = d({}_2f')$, then $V(f_0, f_1, {}_2f) = V(f_0', f_1', {}_2f')$ again by Axiom 8. Second, if $(f_0, f_1, {}_2f)$ and $(f_0', f_1', {}_2f')$ are such that either $(f_0, f_1, {}_2f') \notin \mathcal{F}_0$ or $(f_0', f_1', {}_2f) \notin \mathcal{F}_0$, then $(f_0, f_1) \neq (f_0', f_1')$. So, even if $d({}_2f) = d({}_2f')$, $\tilde{V}(f_0, f_1, d({}_2f))$ can be different from $\tilde{V}(f_0', f_1', d({}_2f'))$.

Consider now (f_1, d''), $(f_1', d'') \in \tilde{\mathcal{Q}}$. There exist $c, c' \in C$, such that $f_t = f_t(c)$ and $f_t' = f_t(c')$ for $t = 0, 1$, and $d({}_2f(c)) = d({}_2f(c')) = d''$. By Lemma 2, without loss, we can assume that ${}_2c = {}_2c'$ so that ${}_2f(c) = {}_2f(c') = {}_2f''$. By Axiom 7(i), then $V(f_0, f_1, {}_2f'') > V(f_0, f_1', {}_2f'')$ iff $f_1 > f_1'$, and property (5.ii) follows from (14).

Finally, $\tilde{V}$ is continuous for the following reasons. For any $(f_1, d) \in \tilde{\mathcal{Q}}$, $\tilde{V}(\cdot, f_1, d) = U(\cdot, {}_1c)$ for any c such that $f_1 = f_1(c)$ and $d = d({}_2f(c))$. Hence, the continuity property of U implies that $\tilde{V}$ is continuous in its first argument. Given any f_0 and $d \in \mathcal{D}$, $\tilde{V}(f_0, \cdot, d) = U(f_0, \cdot, {}_2c)$ for some c such that $f_0(c) = f_0$ and $d = d({}_2f(c))$. Hence, $\tilde{V}$ must take value in a connected interval and, being strictly increasing, it must be continuous in its second argument given f_0 and d. By a similar argument, for any (f_0, f_1), $\tilde{V}(f_0, f_1, \cdot)$ must take values in a connected interval and, being strictly increasing, it must be continuous in its last argument. It follows that $\tilde{V}$ must be continuous on the connected set $X \times \tilde{\mathcal{Q}}$.

□

Now define $\mathcal{Q} = \{(f_1(c), f_2(c), d({}_3f(c))) : c \in C\}$. By an argument similar to that in the proof of Lemma 5, using Lemma 1 with $\pi = \{3\}$, we obtain the following.

[29] Hereafter, "iff" stands for "if and only if."

Lemma 6. *There exists a continuous function* $\overline{V} : X \times \mathcal{Q} \to \mathbb{R}$ *such that, for all* $f \in \mathcal{F}_0$,

$$V(f) = \overline{V}(f_0, f_1, f_2, d({}_3f)), \tag{15}$$

where d *is the function defined in Lemma 2. Moreover, for any* f_1, f_1', f_2, f_2', d', *and* d'' *we have the following:*
(6.i) if (f_1, f_2, d'), $(f_1', f_2, d') \in \mathcal{Q}$, *then* $\overline{V}(f_0, f_1, f_2, d') > \overline{V}(f_0, f_1', f_2, d')$ *iff* $f_1 > f_1'$;
(6.ii) if (f_1, f_2, d'), $(f_1, f_2', d') \in \mathcal{Q}$, *then* $\overline{V}(f_0, f_1, f_2, d') > \overline{V}(f_0, f_1, f_2', d')$ *iff* $f_2 > f_2'$;
(6.iii) if (f_1, f_2, d'), $(f_1, f_2, d'') \in \mathcal{Q}$, *then* $\overline{V}(f_0, f_1, f_2, d') > \overline{V}(f_0, f_1, f_2, d'')$ *iff* $d' > d''$.

Hereafter, for any $c \in C$, let $d_3(c) = d({}_3f(c))$. Also, we say that $c \in C$ induces $(f_0, f_1, f_2, d_3) \in X \times \mathcal{Q}$ if $f_t(c) = f_t$ for $t = 0, 1, 2$ and $d_3(c) = d_3$. Note that the function $\overline{V}$ defines a preference $\bar{\succ}$ on $X \times \mathcal{Q}$; moreover, by definition, for $c, c' \in C$

$$(f_0(c), f_1(c), f_2(c), d_3(c)) \bar{\succ} (f_0(c'), f_1(c'), f_2(c'), d_3(c')) \Leftrightarrow f(c) \succ^* f(c').$$

Lemma 7. *The preference* $\bar{\succ}$ *satisfies the following property (see Definition 4 in Debreu (1960)). Fix any nonempty subset* π *of* $\{0, 1, 2, 3\}$. *Then*

$$(f_0, f_1, f_2, d_3) \bar{\succ} (f_0', f_1', f_2', d_3') \Leftrightarrow (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{d}_3) \bar{\succ} (\hat{f}_0', \hat{f}_1', \hat{f}_2', \hat{d}_3')$$

where $f_t = \hat{f}_t$, $f_t' = \hat{f}_t'$, $d_3 = \hat{d}_3$, *and* $d_3' = \hat{d}_3'$ *if* t *or* 3 *are in* π, *and* $f_t = f_t'$, $\hat{f}_t = \hat{f}_t'$, $d_3 = d_3'$, *and* $\hat{d}_3 = \hat{d}_3'$ *if* t *or* 3 *are not in* π.

Proof. Given π, let π^c be its complement. If $3 \in \pi^c$, then there exist $c, c', \hat{c}, \hat{c}' \in C$ such that, for $t = 0, 1, 2$, $f_t = f_t(c)$, $f_t' = f_t(c')$, $\hat{f}_t' = f_t(\hat{c}')$, $\hat{f}_t = f_t(\hat{c})$, $d({}_3f(c)) = d({}_3f(c'))$, and $d({}_3f(\hat{c})) = d({}_3f(\hat{c}'))$. Then, by Lemma 2, $f_2(c) = f_2(c_0, c_1, c_2, {}_3c')$ and $f_2(\hat{c}) = f_2(\hat{c}_0, \hat{c}_1, \hat{c}_2, {}_3\hat{c}')$. Similarly, by Lemma 5, $f_1(c) = f_1(c_0, c_1, c_2, {}_3c')$ and $f_1(\hat{c}) = f_1(\hat{c}_0, \hat{c}_1, \hat{c}_2, {}_3\hat{c}')$. Therefore, we can take ${}_3c = {}_3c'$ and ${}_3\hat{c} = {}_3\hat{c}'$, so that ${}_3f = {}_3f'$ and ${}_3\hat{f} = {}_3\hat{f}'$.[30] It follows from Lemma 1, that

$$V(f_0, f_1, f_2, {}_3f) > V(f_0', f_1', f_2', {}_3f') \Leftrightarrow V(\hat{f}_0, \hat{f}_1, \hat{f}_2, {}_3\hat{f}) > V(\hat{f}_0', \hat{f}_1', \hat{f}_2', {}_3\hat{f}').$$

Hence, by (15), the result follows.

Suppose $3 \in \pi$. Again, there exist $c, c', \hat{c}, \hat{c}' \in C$, each inducing the respective element of $X \times \mathcal{Q}$—in particular, $d({}_3f(c)) = d({}_3f(\hat{c}))$ and $d({}_3f(c')) = d({}_3f(\hat{c}'))$. Then, by Lemma 2, $f_2(c) = f_2(c_0, c_1, c_2, {}_3\hat{c})$ and $f_2(c') = f_2(c_0', c_1', c_2', {}_3\hat{c}')$. Similarly, by Lemma 5, $f_1(c) = f_1(c_0, c_1, c_2, {}_3\hat{c})$ and $f_1(c') = f_1(c_0', c_1', c_2', {}_3\hat{c}')$. Therefore, we can take ${}_3c = {}_3\hat{c}$ and ${}_3c' = {}_3\hat{c}'$, so that ${}_3f = {}_3\hat{f}$ and

[30] Recall that by Lemma 6, if $(f_0, f_1, f_2, {}_3\tilde{f})$ and $(f_0, f_1, f_2, {}_3\tilde{f}')$ are in $\mathcal{F}_0$ and $d({}_3\tilde{f}) = d({}_3\tilde{f}')$, then $V(f_0, f_1, f_2, {}_3\tilde{f}) = V(f_0, f_1, f_2, {}_3\tilde{f}')$.

${}_3f' = {}_3\hat{f}'$. It follows again from Lemma 1, that

$$V(f_0, f_1, f_2, {}_3f) > V(f'_0, f'_1, f'_2, {}_3f') \Leftrightarrow V(\hat{f}_0, \hat{f}_1, \hat{f}_2, {}_3\hat{f}) > V(\hat{f}'_0, \hat{f}'_1, \hat{f}'_2, {}_3\hat{f}').$$

Hence, by (15), the result follows.

□

Step 3: Let $\mathcal{O}$ be the set of vectors $(f_1(c), f_2(c), d({}_3f(c)))$ for $c \in C$, such that $\underline{u} < u(c_t) < \overline{u}$ for $t = 1, 2$ and $d({}_3f(c)) \in int\mathcal{D}$. It is straightforward to check that $\mathcal{O}$ is nonempty and that $\mathcal{Q}$ is included in the closure of $\mathcal{O}$.[31]

Lemma 8. *For any $(f_1, f_2, d_3) \in \mathcal{O}$, there exists $\eta > 0$ such that the rectangle*

$$\mathcal{R}(f_1, f_2, d_3; \eta) = (f_1 - \eta, f_1 + \eta) \times (f_2 - \eta, f_2 + \eta) \times (d_3 - \eta, d_3 + \eta)$$

lies in $\mathcal{O}$.

Proof. Fix $(f_1, f_2, d_3) \in \mathcal{O}$ and, for the inducing c, let $u_t = u(c_t)$ for $t = 1, 2$. Since $d_3 \in int\mathcal{D}$, there is an interval $(\underline{d}_3, \overline{d}_3) \subset \mathcal{D}$ containing d_3. Since $\underline{u} < u_2 < \overline{u}$, given d_3, there is an interval $(\underline{f}_2(d_3), \overline{f}_2(d_3)) \subset \mathcal{U}$, containing f_2 and spanned by $u_2 \in int\mathcal{I}_u$. Let $\eta' > 0$ be such that $[d_3 - \eta', d_3 + \eta'] \subset (\underline{d}_3, \overline{d}_3)$. By the properties of W in Lemma 2, there exists $\eta' > 0$ such that $\underline{f}_2(d_3) < \underline{f}_2(d_3 + \eta') < f_2$ and $\overline{f}_2(d_3) > \overline{f}_2(d_3 - \eta') > f_2$. Hence, for all $d'_3 \in [d_3 - \eta', d_3 + \eta']$, all $f'_2 \in [f_2 - \varepsilon(\eta'), f_2 + \varepsilon(\eta')]$ are achievable by changing only u_2, where $\varepsilon(\eta') = \min\{f_2 - \underline{f}_2(d_3 + \eta'), \overline{f}_2(d_3 - \eta') - f_2\}$. Since $\underline{u} < u_1 < \overline{u}$, given f_2 and d_3, there is an interval $(\underline{f}_1(f_2, d_3), \overline{f}_1(f_2, d_3)) \subset \mathcal{U}$, containing f_1 and spanned by $u_1 \in int\mathcal{I}_u$. By the properties of $\tilde{V}$ in Lemma 5, there exist $\eta'' > 0$ and $\varepsilon'' > 0$ such that $[d_3 - \eta'', d_3 + \eta''] \subset (\underline{d}_3, \overline{d}_3)$, $[f_2 - \varepsilon'', f_2 + \varepsilon''] \subset (\underline{f}_2(d_3), \overline{f}_2(d_3))$, and $\underline{f}_1(f_2, d_3) < \underline{f}_1(f_2 + \varepsilon'', d_3 + \eta'') < f_1$ and $\overline{f}_1(f_2, d_3) > \overline{f}_1(f_2 - \varepsilon'', d_3 - \eta'') > f_1$. Hence, for all $(f''_2, d''_3) \in [f_2 - \varepsilon'', f_2 + \varepsilon''] \times [d_3 - \eta'', d_3 + \eta'']$, all $f''_1 \in [f_1 - \delta(\varepsilon'', \eta''), f_1 + \delta(\varepsilon'', \eta'')]$ are achievable by changing only u_1, where $\delta(\varepsilon'', \eta'') = \min\{f_1 - \underline{f}_1(f_2 + \varepsilon'', d_3 + \eta''), \overline{f}_1(f_2 - \varepsilon'', d_3 - \eta'') - f_1\}$. Let $\hat{\eta} = \min\{\eta', \eta''\}$, $\varepsilon = \min\{\varepsilon(\hat{\eta}), \varepsilon''\}$, and $\delta = \delta(\varepsilon, \hat{\eta})$. Noting that $\varepsilon(\hat{\eta}) \geq \varepsilon(\eta')$ and letting $\eta = \min\{\hat{\eta}, \varepsilon, \delta\}$, we have that all (f'_1, f'_2, d'_3) in $\mathcal{R}(f_1, f_2, d_3; \eta)$ are induced by some $c \in C$ and belong to $\mathcal{O}$.

□

[31] To see that $\mathcal{O} \neq \varnothing$, consider any constant $c' \in C$ such that $\underline{u} < u(c'_0) < \overline{u}$. By changing c'_3 so that $u(c_3)$ varies continuously in an open interval around $u(c'_3)$, by continuity of U we can continuously span an open interval around $f_3(c')$. By Axiom 7(i), this variation in c_3 leads to variations in $f_2(c)$, which must span an open interval around $f_2(c')$, again by continuity of U. Since we are not changing c'_2, by Lemma 2, $d_3(c)$ must change in an open interval around $d_3(c')$. Finally, by Lemma 5, $f_1(c)$ must also vary continuously in an open interval around $f_1(c')$. To see that $\mathcal{Q} \subset cl\mathcal{O}$, notice that any point of $\mathcal{Q}$ induced by some $c \in C$ can be approximated, by slightly modifying c, by a c' such that $d_3(c') \in int\mathcal{D}$ and $u(c'_t) \in int\mathcal{I}_u$ for $t = 1, 2$, i.e., a point in $\mathcal{O}$.

Lemma 9. *$\mathcal{O}$ is connected.*

Proof. We will show that $\mathcal{O}$ is path connected and hence connected. Take any (f_1', f_2', d_3'), $(f_1'', f_2'', d_3'') \in \mathcal{O}$ with inducing streams $c', c'' \in C$. By definition, $u(c_t')$, $u(c_t'') \in int\mathcal{I}_u$ for $t = 1, 2$ and $d_3', d_3'' \in int\mathcal{D}$. Since $\mathcal{D}$ is an interval, we can vary consumption from $t = 3$ onward, creating a path from ${}_3c'$ to ${}_3c''$ so as to cover the interval between d_3' and d_3''. Along this path d_3 remains in $int\mathcal{D}$; moreover, by Lemma 2, f_2 varies covering an interval between f_2' and $f_2(c_0', c_1', c_2', {}_3c'')$, and by Lemma 5, f_1 varies covering an interval between f_1' and $f_1(c_0', c_1', c_2', {}_3c'')$. Since c_1' and c_2' are unchanged, all (f_1, f_2, d_3) along the path are in $\mathcal{O}$. Now fix ${}_3c = {}_3c''$ and vary c_2 to create a path from c_2' to c_2'' so as to cover the interval between $u(c_2')$ and $u(c_2'')$. Along this path $u(c_2)$ remains in $int\mathcal{I}_u$; moreover, by Lemma 2, f_2 varies covering the interval between $f_2(c_0', c_1', c_2', {}_3c'')$ and $f_2(c_0', c_1', c_2'', {}_3c'')$, and by Lemma 5, f_1 varies covering an interval between $f_1(c_0', c_1', c_2', {}_3c'')$ and $f_1(c_0', c_1', c_2'', {}_3c'')$. Since c_1' is unchanged, again all (f_1, f_2, d_3) along this second path are in $\mathcal{O}$. Finally, fix ${}_2c = {}_2c''$ and vary c_1 to create a path from c_1' to c_1'' so as to cover the interval between $u(c_1')$ and $u(c_1'')$. Along this path $u(c_1)$ remains in $int\mathcal{I}_u$; moreover, by Lemma 2, f_1 varies covering the interval between $f_1(c_0', c_1', c_2'', {}_3c'')$ and $f_1(c_0', c_1'', c_2'', {}_3c'')$. Since c_2'' is unchanged, again all (f_1, f_2, d_3) along this third path are in $\mathcal{O}$. The three paths together form a connected path from (f_1', f_2', d_3') to (f_1'', f_2'', d_3'') which never leaves $\mathcal{O}$.

□

We are now ready to obtain an additive representation of $\bar{\succ}$, relying on Debreu (1960).

Lemma 10. *The preference $\bar{\succ}$ over $X \times \mathcal{Q}$ can be represented by an additive function*

$$V^0(f_0, f_1, f_2, d_3) = \hat{u}(f_0) + a(f_1) + b(f_2) + \zeta(d_3),$$

where $\hat{u}$, a, b, and ζ are continuous, and a, b, ζ are strictly increasing on $\mathcal{U}$.

Proof. We first show that $\bar{\succ}$ has an additive representation over $X \times \mathcal{O}$. By continuity, we then extend this representation to $X \times \mathcal{Q}$. The representation of $\bar{\succ}$ over $X \times \mathcal{Q}$ immediately implies the desired representation of $\succ^*$ on $\mathcal{F}_0$.

The set $\mathcal{O}$ may be expressed as a countable union of open rectangles $\{\mathcal{R}^i\}_{i\in\mathbb{N}}$ of the form in Lemma 8, and such that for any j there is an $i < j$ such that $\mathcal{R}^i \cap \mathcal{R}^j \neq \varnothing$. To construct $\{\mathcal{R}^i\}_{i\in\mathbb{N}}$, proceed as follows. Let $\{\overline{\mathcal{B}}^n\}_{n=1}^{\infty}$ be the sequence of closed balls of radius n centered at the origin in $\mathbb{R}^3$. Then, let

$$\mathcal{K}^n = \{o \in \mathcal{O} \mid o \in \overline{\mathcal{B}}^n, \mathcal{B}^{1/n}(o) \subset \mathcal{O}\},$$

where $\mathcal{B}^{1/n}(o)$ is the open ball of radius $1/n$ centered at point o. For each n, $\mathcal{K}^n$ is compact[32]

[32] $\mathcal{K}^n$ is clearly bounded. Consider any sequence $\{o^m\} \subset \mathcal{K}^n$ converging to o'. Since $\overline{\mathcal{B}}^n$ is closed,

and the increasing sequence $\{\mathcal{K}^n\}_{n=1}^{\infty}$ converges to $\mathcal{O}$. So, each $\mathcal{K}^n$ can be covered by finitely many rectangles of the form in Lemma 8. Since $\mathcal{K}^n \subset \mathcal{K}^{n+1}$, when moving from $\mathcal{K}^n$ to $\mathcal{K}^{n+1}$, one can cover $\mathcal{K}^{n+1}$ by simply adding rectangles to those used to cover $\mathcal{K}^n$. Without loss, any added rectangle contains a point with rational coordinates not contained in other rectangles, so that the list of rectangles needed to cover $\mathcal{O}$, denoted by $\{\mathcal{R}^i\}_{i\in\mathbb{N}}$, is countable. Finally, since $\mathcal{O}$ is connected, each $\mathcal{R}^j$ must intersect at least another $\mathcal{R}^i$. For simplicity, we can relabel the rectangles so that, for each j, we have $\mathcal{R}^j \cap \mathcal{R}^i \neq \varnothing$ for some $i < j$.

For any $\mathcal{R}^i$, Lemmas 4 and 7 guarantee that the hypotheses of Debreu's (1960) Theorem 3 are satisfied on $X \times \mathcal{R}^i$. Therefore, $\bar{\succsim}$ may be expressed over each $X \times \mathcal{R}^i$ as

$$V^i(f_0, f_1, f_2, d_3) = \hat{u}^i(f_0) + a^i(f_1) + b^i(f_2) + \zeta^i(d_3),$$

for functions $\hat{u}^i, a^i, b^i$, and ζ^i that are continuous and, except for $\hat{u}^i$, strictly increasing by the properties of $\bar{V}$ which induces $\bar{\succsim}$.[33]

By construction, $\mathcal{R}^0$ and $\mathcal{R}^1$ have a nonempty open intersection. Over $X \times (\mathcal{R}^0 \cap \mathcal{R}^1)$ representations V^0 and V^1 must be positive affine transformations of each other (Debreu's (1960) Theorem 3). So there exist constants $\rho > 0$ and $\chi \in \mathbb{R}$ such that, on $X \times (\mathcal{R}^0 \cap \mathcal{R}^1)$,

$$\hat{u}^0(f_0) = \rho\hat{u}^1(f_0) + \chi, \quad a^0(f_1) = \rho a^1(f_1), \qquad b^0(f_2) = \rho b^1(f_2), \quad \zeta^0(d_3) = \rho\zeta^1(d_3).$$

Using these conditions, we can extend $\hat{u}^0, a^0, b^0$, and ζ^0 to the set $X \times (\mathcal{R}^0 \cup \mathcal{R}^1)$. Indeed, each function a^i, b^i, and ζ^i is defined on $\mathcal{R}^i_k$ which denotes the projection of $\mathcal{R}^i$ on the k^{th} dimension of $\mathcal{Q}$. Consider a^0. By extending a^0 over $\mathcal{R}^1_1 \setminus \mathcal{R}^0_1$ using a^1, the resulting function a^0 is well defined and continuous on $\mathcal{R}^0_1 \cup \mathcal{R}^1_1$. By a similar reasoning for b^0 and ζ^0, the function V^0 can be extended to $X \times (\mathcal{R}^0_1 \cup \mathcal{R}^1_1) \times (\mathcal{R}^0_2 \cup \mathcal{R}^1_2) \times (\mathcal{R}^0_3 \cup \mathcal{R}^1_3)$. Since this product includes $X \times \mathcal{R}^0 \cup \mathcal{R}^1$, the function V^0 is, in particular, well defined and continuous on it.

Finally, since for each $j > 0$ we have $\mathcal{R}^j \cap \mathcal{R}^i \neq \varnothing$ for some $i < j$, we can extend by induction representation V^0 from $X \times \mathcal{R}^0$ to $X \times (\cup_{i\in\mathbb{N}}\mathcal{R}^i) = X \times \mathcal{O}$, in countably many steps. Notice that the functions a, b, and ζ (we henceforth omit the superscript '0') entering the formula of V^0 are defined, through the induction, over the respective projections of $\mathcal{O}$.

Since any point of $X \times \mathcal{O}$ is contained in $X \times \mathcal{R}^i$ for some $i \in \mathbb{N}$, V^0 and its components $\hat{u}$, a, b, and ζ are continuous over $X \times \mathcal{O}$. Moreover, V^0 represents $\bar{\succsim}$ on $X \times \mathcal{O}$. To see this, we need to check that for any (f_0', f_1', f_2', d_3') and $(f_0'', f_1'', f_2'', d_3'')$ in $X \times \mathcal{O}$, $V^0(f_0', f_1', f_2', d_3') > V^0(f_0'', f_1'', f_2'', d_3'')$ iff $(f_0', f_1', f_2', d_3') \bar{\succ} (f_0'', f_1'', f_2'', d_3'')$. Note that (f_1', f_2', d_3') and (f_1'', f_2'', d_3'') must both belong to some $\mathcal{K}^n$ in the previous construction. Since V^0 represents $\bar{\succsim}$ on $X \times \mathcal{K}^n$, it

$o' \in \overline{\mathcal{B}}^n$. There remains to show that $\mathcal{B}^{1/n}(o') \subset \mathcal{O}$. Let o'' be any point such that $||o' - o''|| = r < 1/n$. Then $||o'' - o^m|| \leq r + ||o' - o^m||$. So, for m large enough, $o'' \in \mathcal{B}^{1/n}(o^m)$ and hence $o'' \in \mathcal{O}$.

[33] While Debreu's theorem requires that the preference domain be a Cartesian product, it does not assume compactness of the sets forming the product.

ranks (f'_0, f'_1, f'_2, d'_3) and $(f''_0, f''_1, f''_2, d''_3)$ correctly, which proves the claim.

It remains to show that V^0 can be extended to the entire domain $X \times \mathcal{Q}$, additively, and that it represents $\bar{\succ}$ over this domain. We first show that V^0 can be extended to a continuous function over $X \times \mathcal{Q}$. Recall that $\overline{V}$ is continuous and represents $\bar{\succ}$ over $X \times \mathcal{Q}$—and hence over $X \times \mathcal{O}$. So, there exists a strictly increasing map $\phi : Y \to Y^0$ such that $V^0 = \phi \circ \overline{V}$, where Y^0 and Y are the ranges of V^0 and $\overline{V}$ on $X \times \mathcal{O}$. Y^0 and Y are intervals of $\mathbb{R}$ because $X \times \mathcal{O}$ is connected and V^0 and $\overline{V}$ are continuous over this domain. Since ϕ is strictly increasing, it must be continuous on its domain, otherwise it would not cover Y^0. Let $\overline{Y}$ be the range of $\overline{V}$ over $X \times \mathcal{Q}$. Since $X \times \mathcal{Q} \subset cl(X \times \mathcal{O})$ and $\overline{V}$ is continuous, $\overline{Y}$ contains at most two more points than Y (its boundaries), and this may occur only when the relevant boundaries are finite. One can extend ϕ to these points, whenever applicable, by taking the limit of ϕ: for example, if $\bar{y}$ denotes the upper bound of $\overline{Y}$ and $\bar{y} \notin Y$, one may define $\phi(\bar{y})$ as $\lim_{y \uparrow \bar{y}} \phi(y)$.[34] Finally, we can extend V^0 to $X \times \mathcal{Q}$ by letting $V^0 = \phi \circ \overline{V}$ over this domain. By construction, V^0 is continuous as the composition of continuous functions.

Next, we show that this extension of V^0 to $X \times \mathcal{Q}$ still obeys the additive representation obtained on $X \times \mathcal{O}$ in terms of $\hat{u}, a, b$ and ζ. We first show that a, b, and ζ can be extended on the relevant projections of $\mathcal{Q}$ (not just of $\mathcal{O}$). Since $\mathcal{O}$ is connected and $\mathcal{Q} \subset cl\mathcal{O}$, the extension is only needed (possibly) at the two boundaries of $\mathcal{D}$ for ζ, and at the boundaries of $\mathcal{U}$ for a and b; these extensions are necessary only if these boundaries are achieved by some $(f_1, f_2, d_3) \in \mathcal{Q}$.

To extend ζ, suppose that there is an $(f_1, f_2, d_3) \in \mathcal{Q}$ such that d_3 is the upper bound of $\mathcal{D}$—the other case follows similarly. Without loss, we can choose $f_1,\ f_2 \in int\mathcal{U}$.[35] By perturbing c_3, we can then construct a sequence $\{(f_1^n, f_2^n, d_3^n)\}$ such that f_1^n and f_2^n are in some compact $K \subset int\mathcal{U}$ and $d_3^n \in int\mathcal{D}$ for all n, and $d_3^n \to d_3$. By construction, each $(f_1^n, f_2^n, d_3^n) \in \mathcal{O}$. Fixing some f_0, the sum $\hat{u}(f_0) + a(f_1^n) + b(f_2^n) + \zeta(d_3^n)$ is well defined and equal to $V^0(f_0, f_1^n, f_2^n, d_3^n)$ for each n. Moreover, possibly moving to subsequences, $f_1^n \to \hat{f}_1$ and $f_2^n \to \hat{f}_2$ for some $\hat{f}_1, \hat{f}_2 \in K$. Since a and b are continuous over K, $a(f_1^n)$ and $b(f_2^n)$ converge on these subsequences. Therefore, $\zeta(d_3)$

[34] One can show that for $\bar{y} \in \overline{Y} \setminus Y$, $\lim_{y \uparrow \bar{y}} \phi(y)$ must be finite. Suppose not: First, there exist i) $\bar{s} = (\bar{f}_0, \bar{f}_1, \bar{f}_2, \bar{d}) \in X \times \mathcal{Q}$ such that $\bar{V}(\bar{s}) = \bar{y}$, which means that the agent prefers $\bar{s}$ to any other stream; and ii) a sequence $s^n = (f_0^n, f_1^n, f_2^n, d^n) \in X \times \mathcal{O}$ that converges to $\bar{s}$, and such that $V^0(s^n)$ diverges to $+\infty$. Because V^0 is additive, this means that there must be at least one sequence, among $\hat{u}(f_0^n)$, $a(f_1^n)$, $b(f_2^n)$, and $\zeta(d^n)$, with a subsequence diverging to $+\infty$. For example, suppose that d^n is such that $\zeta(d^n)$ diverges to $+\infty$. Then, for any stream c such that $d({}_3f(c)) = \bar{d}$, we have $\bar{V}(f_0(c), f_1(c), f_2(c), \bar{d}(c)) = \bar{y}$. Indeed, fix any c_0 and c_t's such that $\underline{u} < u(c_t) < \bar{u}$ for $t \in 1, 2$. Choosing the sequence of continuation streams $(c_3, \ldots)$ corresponding to the sequence of d^n converging to $\bar{d}$, V^0 evaluated at those streams (and the fixed c_0, c_1, c_2) must diverge to $+\infty$. This implies that $\bar{V}$ converges to $\bar{y}$ for that sequence. By continuity of $\bar{V}$ over its entire domain, this implies that when choosing $(c_3, \ldots)$ such that $d({}_3f(c)) = \bar{d}$, we have $\bar{V}(f_0(c), f_1(c), f_2(c), d({}_3f(c))) = \bar{y}$, regardless of the values of c_0, c_1, c_2. This, however, violates the fact that preferences are strictly increasing in $u(c_0)$ (Lemma 2), a contradiction. A similar contradiction can be derived if instead $\hat{u}(f_0^n)$, or $a(f_1^n)$, or $b(f_2^n)$ has a subsequence diverging to $+\infty$. This shows that necessarily Y_0 is bounded above whenever $\bar{y} \in \bar{Y} \setminus Y$. By a similar argument for the lower bound, we conclude that ϕ is bounded at any boundary for which it needs to be extended.

[35] This can be achieved by changing c_1 and c_2 of the stream inducing (f_1, f_2, d_3), without affecting d_3.

is well defined as the difference $V^0(f_0, \hat{f}_1, \hat{f}_2, d_3) - \hat{u}(f_0) - a(\hat{f}_1) - b(\hat{f}_2)$, because V^0 has already been extended to $(f_0, \hat{f}_1, \hat{f}_2, d_3)$. Moreover, since V^0 was extended continuously over $X \times \mathcal{Q}$, ζ must also be continuous at d_3.

We can similarly extend b to the boundary of $\mathcal{U}$, whenever needed. To see this, take any $(f_1, f_2, d_3) \in \mathcal{Q}$ such that f_2 lies at a boundary of $\mathcal{U}$, say $f_2 = \bar{\nu}$—again, the other case follows similarly. Moreover, we can choose c_1 in the inducing stream $c \in C$ so that $f_1 \in int\mathcal{U}$. By perturbing c_2, we can build a sequence $\{(f_1^n, f_2^n, d_3)\}$ such that f_1^n is in a compact $K \subset int\mathcal{U}$ and $f_2^n \in int\mathcal{U}$ for all n, and $f_2^n \to \bar{\nu}$. Possibly taking a subsequence such that $f_1^n \to \hat{f}_1$ for some $\hat{f}_1 \in K$, we obtain a well define limit for V^0, a, and ζ, from which we can obtain the value of $b(\bar{\nu})$. The argument for a is identical.

In conclusion, the function $\hat{u}(\cdot) + a(\cdot) + b(\cdot) + \zeta(\cdot)$ is equal to V^0 over the entire set $X \times \mathcal{Q}$, and represents $\bar{\succ}$ over this domain.

$\square$

Step 4: By Lemma 1 with $\pi = \{1, 2, 3\}$, for any $f_0 \in X$, the induced preference $\succ^*_{-0}$ on $\mathcal{F}$ is independent of f_0. By Lemma 10, we can conclude that $\succ^*_{-0}$ has a representation

$$V^*_{-0}(f_1, f_2, {}_3f) = a(f_1) + b(f_2) + \zeta(d({}_3f)). \tag{16}$$

Note that Axiom 8 holds for any f_0. So if $f_1 = f_1'$, $(f_1, f_2, {}_3f) \succsim^*_{-0} (f_1', f_2', {}_3f')$ iff $(f_2, f_3, {}_4f) \succsim^*_{-0} (f_2', f_3', {}_4f')$.

Lemma 11. *There exist $\alpha > 0$, $\xi \in \mathbb{R}$, and $G : \mathcal{U} \to \mathbb{R}$ continuous and strictly increasing such that, for any finite $T \geq 2$ any $f \in \mathcal{F}$,*

$$V^*_{-0}(f) = \sum_{t=1}^{T} \alpha^t G(f_t) + \alpha^T \tilde{d}({}_{T+1}f) + \xi \sum_{t=0}^{T-2} \alpha^t. \tag{17}$$

Proof. Consider again $\mathcal{R}^0$ in the proof of Lemma 10. By definition of a rectangle, if $(f_1, f_2, {}_3f)$ and $(f_1', f_2', {}_3f')$ are such that $(f_1, f_2, d({}_3f))$, $(f_1', f_2', d({}_3f')) \in \mathcal{R}^0$, then all $\hat{f}_1 \in \mathcal{R}^0_1$ are feasible with both $(f_2, {}_3f)$ and $(f_2', {}_3f')$. By the stationarity property of $\succ^*_{-0}$, we have

$$a(\hat{f}_1) + b(f_2) + \zeta(d({}_3f)) \geq a(\hat{f}_1 Let) + b(f_2') + \zeta(d({}_3f'))$$

iff

$$a(f_2) + b(f_3) + \zeta(d({}_4f)) \geq a(f_2') + b(f_3') + \zeta(d({}_4f')).$$

Hence, since additive representations are unique up to positive affine transformations, for all $(f_2, {}_3f)$ such that $(f_1, f_2, d({}_3f)) \in \mathcal{R}^0$,

$$\alpha(a(f_2) + b(f_3) + \zeta(d({}_4f))) + \xi = b(f_2) + \zeta(d({}_3f)) \tag{18}$$

for some $\alpha > 0$ and $\xi \in \mathbb{R}$.

The argument used for $\mathcal{R}^0$ can be equivalently applied to any $\mathcal{R}^i$ in the covering $\{\mathcal{R}^i\}_{i\in\mathbb{N}}$ of $\mathcal{O}$. Moreover, since for each $j > 0$ we have $\mathcal{R}^j \cap \mathcal{R}^i \neq \varnothing$ for some $i < j$, it is clear that the α in (18) must be the same for all $f \in \mathcal{F}$ such that $(f_1, f_2, d({}_3f)) \in \mathcal{O}$. That (18) must hold for all $f \in \mathcal{F}$ is implied by the following two observations. First, if $c \in C$ induces $(f_1, f_2, d_3) \in \mathcal{O}$, it imposes no restriction on $d({}_4f(c))$, which can take any value in $\mathcal{D}$—hence ${}_4f$ can take any value in $\mathcal{F}$. To see this, recall that for $f \in \mathcal{F}$ we defined $d(f) = V(\hat{f}_0, f)$ for some $\hat{f}_0 \in X$, and $V(\hat{f}_0, f) = \tilde{V}(\hat{f}_0, f_1, d({}_2f))$ by Lemma 5. So, since $\tilde{V}$ is strictly increasing in its second and third argument, the condition $d_3(c) \in int\mathcal{D}$ only implies $f_3(c) \in int\mathcal{U}$, but $d({}_4f(c))$ can be at the boundary of $\mathcal{D}$. Therefore, (18) already holds for any value of ${}_4f \in \mathcal{F}$. Second, suppose that f is such that $(f_1, f_2, d({}_3f))$ is at boundary of $\mathcal{Q}$. Take a sequence $\{f^n\}$ such that $(f_1^n, f_2^n, d({}_3f^n)) \in \mathcal{O}$ for all n and converges to $(f_1, f_2, d({}_3f))$. The sequence can be chosen so that ${}_4f$ is fixed: perturbing only c_1, c_2, and c_3 is enough to guarantee that we are in $\mathcal{O}$. Now recall that the functions a, b, and ζ are continuous by Lemma 10. Then, the right-hand side of (18) converges, as do the first two terms of the left-hand side. The last term is constant and equal to $\zeta(d({}_4f))$, so it converges trivially. Therefore (18) holds everywhere.

We conclude that, for all $f \in \mathcal{F}$,

$$V^*_{-0}(f_1, f_2, {}_3f) = a(f_1) + \xi + \alpha V^*_{-0}(f_2, f_3, {}_4f).$$

Therefore, using this condition recursively and (16), for any $f \in \mathcal{F}$ and finite $T > 2$, we have

$$V^*_{-0}(f_1, f_2, {}_3f) = \sum_{t=0}^{T-1} \alpha^t a(f_{t+1}) + \alpha^{T-1}(b(f_{T+1}) + \zeta(d({}_{T+2}f))) + \xi \sum_{t=0}^{T-2} \alpha^t.$$

The result then follows by defining $G = \alpha^{-1}a$ and $\tilde{d}(\cdot) = \alpha^{-1}(b(\cdot) + \zeta(d(\cdot)))$.

□

By Lemma 11, for any finite $T \geq 2$, we can represent $\succ$ for streams c as

$$\overline{U}(c) = \hat{u}(c_0) + \sum_{t=1}^{T} \alpha^t G(U({}_tc)) + \alpha^T \tilde{d}(U({}_{T+1}c), U({}_{T+2}c), \ldots) + \xi \sum_{t=0}^{T-2} \alpha^t. \qquad (19)$$

The next two technical lemmas will be useful to complete the proof of our theorem.

Lemma 12. *For any constant streams $c, c' \in C$, $c \succ c'$ iff $\hat{u}(c_0) > \hat{u}(c_0')$.*

Proof. Suppose $\hat{u}(x) > \hat{u}(y)$ and consider $c = (x, x, \ldots)$ and $\hat{c} = (x, y, \ldots)$. For any $t \geq 0$ and $c'' \in C$, define $c^t = (c_0, \ldots, c_t, c'')$ and $\hat{c}^t = (\hat{c}_0, \ldots, \hat{c}_t, c'')$. For $t = 0$, we have $\overline{U}(c^t) = \overline{U}(\hat{c}^t)$. For any $t > 0$, using (19), we first have $\overline{U}({}_tc^t) > \overline{U}({}_t\hat{c}^t)$. Then, using again (19) backward recursively and monotonicity of G, we conclude that $\overline{U}(c^t) \geq \overline{U}(\hat{c}^t)$. Since this is true for any $t \geq 0$ and $c'' \in C$, Axiom 7(ii) implies $c \succsim \hat{c}$. Now note that, again by (19), $\hat{c} \succ (y, y, \ldots)$. Hence, by Axiom 1, $c \succ (y, y, \ldots)$.

Now suppose $\hat{u}(x) = \hat{u}(y)$ and consider $c = (x, x, \ldots)$ and $\hat{c} = (y, y, \ldots)$. For any t and $c'' \in C$, define c^t and $\hat{c}^t$ as before. Using again (19) backward recursively and the fact that G is a function, we conclude that $\overline{U}(c^t) = \overline{U}(\hat{c}^t)$. Since this is true for any t and $c'' \in C$, Axiom 7(ii) implies $c \sim c'$.

□

Lemma 13. *For any $c \in C$, there exists $x \in X$ such that $c \sim (x, x, \ldots)$.*

Proof. By Lemma 19 in Appendix B, for any $c \in C$, there exists $y \in X$ such that $c \sim (c_0, y, y, \ldots)$. Suppose $(c_0, y, y, \ldots) \not\sim (y, y, \ldots)$. If $(c_0, y, y, \ldots) \succ (y, y, \ldots)$, then $\hat{u}(c_0) > \hat{u}(y)$. Let $\hat{c} = (c_0, c_0, \ldots)$ and $\tilde{c} = (c_0, y, y, \ldots)$. For any $t \geq 0$ and any $c'' \in C$, consider $\hat{c}^t = (\hat{c}_0, \ldots, \hat{c}_t, c'')$ and $\tilde{c}^t = (\tilde{c}_0, \ldots, \tilde{c}_t, c'')$. We have $\hat{c}^t \succsim \tilde{c}^t$. Indeed, for $t = 0$, $\hat{c}^t = \tilde{c}^t$. For $t > 0$, we can proceed using (19). Since $\hat{u}(c_0) > \hat{u}(y)$, $U({}_t\hat{c}^t) > U({}_t\tilde{c}^t)$. For $s < t$, since $\hat{u}(\hat{c}^t_s) \geq \hat{u}(\tilde{c}^t_s)$ and G is strictly increasing, we have $U({}_s\hat{c}^t) \geq U({}_s\tilde{c}^t)$. By Axiom 7(ii), we then have $\hat{c} \succsim \tilde{c}$ and hence $(c_0, c_0, \ldots) \succsim c \succ (y, y, \ldots)$. Since X is connected, by Axiom 2, there exists $x \in X$ such that $(x, x, \ldots) \sim c$. The case $(c_0, y, y, \ldots) \prec (y, y, \ldots)$ follows similarly.

□

We can now prove that $\alpha < 1$.

Lemma 14. $\alpha < 1$.

Proof. Consider consumption streams that are constant from $t = 3$ onward. Then f_t is constant for $t \geq 3$. So we can write $d({}_3f) = d({}_4f) = e(f_3)$ in (18) and thus obtain

$$(1 - \alpha)e(f_3) = \alpha b(f_3) + \alpha a(f_2) - b(f_2) + \xi.$$

First, note that $f_3 > f'_3$ implies $e(f_3) > e(f'_3)$. By Lemma 12, $f_3 > f'_3$ implies $u(c_3) > u(c'_3)$. Define $c = (c_3, c_3, \ldots)$ and $c' = (c_3, c'_3, c'_3, \ldots)$. Replicating the argument in the proof of Lemma 13, we have $c \succsim (c_3, c_3, c'_3, c'_3, \ldots)$. Moreover, by Axiom 7(i), $(c_3, c_3, c'_3, c'_3, \ldots) \succ c'$. Then, by Axiom 1 and Lemma 2, $W(u(c_3), d(f_3, f_3, \ldots)) > W(u(c_3), d(f'_3, f'_3, \ldots))$, which holds iff $d(f_3, f_3, \ldots) > d(f'_3, f'_3, \ldots)$.

Second, we can find $\hat{c}, \tilde{c} \in C$, constant from $t = 3$ onward, such that $f_2(\hat{c}) = f_2(\tilde{c})$ and $f_3(\hat{c}) > f_3(\tilde{c})$. Consider $x, y \in X$ with $u(x) > u(y)$ and the streams $(x, y, y, \ldots)$ and $(y, x, x, \ldots)$. By the previous argument based on Axiom 7(ii), $(x, x, x, \ldots) \succ (x, y, y, \ldots)$. If $(x, y, y, \ldots) \succsim (y, x, x, \ldots)$, then by (19) and continuity of $\hat{u}$ there exists $z \in X$ such that $(x, y, y, \ldots) \sim (z, x, x, \ldots)$. In this case, let $\hat{c} = (c_0, c_1, z, x, x, \ldots)$. If $(x, y, y, \ldots) \prec (y, x, x, \ldots)$, then by

Axiom 2 there exists $w \in X$ such that $(x, y, y, \ldots) \sim (y, w, w, \ldots)$. Moreover, $u(w) > u(y)$. Otherwise, since $(y, y, y, \ldots) \succsim (y, w, w, \ldots)$ for $u(y) \geq u(w)$ (again by the same argument as before), we would have $(x, y, y, \ldots) \succ (y, w, w, \ldots)$ by (19) and Axiom 1. In this case, let $\hat{c} = (c_0, c_1, y, w, w, \ldots)$. Finally, let $\tilde{c} = (c_0, c_1, x, y, y, \ldots)$.

To conclude the proof, note that for $c \in \{\hat{c}, \tilde{c}\}$, $(1 - \alpha)e(f_3(c)) = \alpha b(f_3(c)) + \xi'$ for some constant ξ'. Since b and e are strictly increasing, we must have $\alpha < 1$.

□

Note that, if c is constant from any $T \geq 3$ onward, by Lemma 14 and (18)

$$\tilde{d}(U({}_Tc), U({}_{T+1}c), \ldots) = \frac{\alpha}{1-\alpha} G(U({}_Tc)) + \frac{\xi}{\alpha(1-\alpha)}.$$

So, for eventually constant streams, we can write

$$\overline{U}(c) = \hat{u}(c_0) + \sum_{t=1}^{T} \alpha^t G(U({}_tc)) + \frac{\alpha^{T+1}}{1-\alpha} G(U({}_{T+1}c)) + \frac{1 + \alpha(1 - \alpha^{T-1})}{\alpha(1-\alpha)} \xi. \tag{20}$$

Lemma 15. *G is bounded on $\mathcal{U}$.*

Proof. By Axiom 3, V^*_{-0} is finite for all $c \in C$. Suppose that G is unbounded above—the other case follows similarly. Then, for each $r \in \mathbb{R}_{++}$, there must be a stream c^r with utility U^r such that $G^r \equiv G(U^r) \geq r$. Moreover, for $r > r'$, we can choose c^r and $c^{r'}$ so that $G^r > G^{r'}$, relying on continuity of G and connectedness of $\mathcal{U}$. By Lemma 13, for each r we can also let c^r be constant. As a preliminary observation, note the following: given $r' > r$, a stream c that equals c^r for the first k periods and $c^{r'}$ forever after must satisfy $G(U(c)) \geq r$. This is because, by definition, $U({}_tc) > U({}_tc^r)$ for $t \geq k$; then, by monotonicity of G and using (20) backward recursively, we have $U({}_tc) > U({}_tc^r)$ for $0 \leq t < k$.

Now construct stream $\hat{c}$ as follows. For some $M > 1$ and each $t > 0$, consider the constant stream $c^{(M/\alpha)^t}$ with the property $\alpha^t G^{(M/\alpha)^t} \geq M^t$. Then, let $\hat{c}_0$ be such that $\underline{u} < \hat{u}(\hat{c}_0) < \overline{u}$ and, for each $t > 1$, let $\hat{c}_t = c_t^{(M/\alpha)^t}$. Now, for any $T > 0$, let c^T be equal to $\hat{c}$ up to T and to $c^{(M/\alpha)^T}$ thereafter. Using (20), we have

$$\begin{aligned}
\overline{U}(c^T) =& \hat{u}(\hat{c}_0) + \sum_{t=1}^{T-1} \alpha^t G(U({}_tc^T)) + \frac{\alpha^T}{1-\alpha} G^{(M/\alpha)^T} + \frac{1 + \alpha(1 - \alpha^{T-2})}{\alpha(1-\alpha)} \xi \\
\geq& \hat{u}(\hat{c}_0) + \sum_{t=1}^{T-1} M^t + \frac{1}{1-\alpha} M^T + \frac{1 + \alpha(1 - \alpha^{T-2})}{\alpha(1-\alpha)} \xi,
\end{aligned}$$

where the inequality follows by recursively applying our preliminary observation. Note that the lower bound on $\overline{U}(c^T)$ goes to $+\infty$ as $T \to \infty$.

Now fix any T and c^T. To simplify notation, let $\tilde{c} = c^T$. Using Axiom 7(ii), we have $\overline{U}(\hat{c}) \geq \overline{U}(\tilde{c})$. To see this, consider any $t \geq 0$ and $c'' \in C$, and let $\hat{c}^t = (\hat{c}_0, \ldots, \hat{c}_t, c'')$ and $\tilde{c}^t = (\tilde{c}_0, \ldots, \tilde{c}_t, c'')$. For $t \leq T$, we have $\hat{c}^t \sim \tilde{c}^t$ because the two streams are identical. For $t > T$, we first have that $u(\hat{c}_s) > u(\tilde{c}_s)$ for $T < s \leq t$ by Lemma 12. Hence, $\overline{U}(\hat{c}_t, c'') > \overline{U}(\tilde{c}_t, c'')$. Second, using again monotonicity of G and (17) recursively, we conclude $\overline{U}(\hat{c}^t) \geq \overline{U}(\tilde{c}^t)$. By Axiom 7(ii), we then have the claimed property.

It follows that, for any T, $\overline{U}(\hat{c}) \geq \overline{U}(c^T)$ and hence, since $\hat{u}(\hat{c}_0)$ is bounded by assumption, $V^*_{-0}(f(\hat{c}))$ must be infinite, violating Axiom 3.

□

Lemma 16. *For any $c \in C$, $\overline{U}(c) = \hat{u}(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_t c))$.*

Proof. Again by Axiom 3, V^*_{-0} is finite for all $c \in C$. Using (17) for any finite T and observing that ${}_T f$ can take any value in $\mathcal{F}$, we conclude that the function $\tilde{d}$ must be finite because G is bounded. The result then follows by letting $T \to \infty$, relying on $\alpha < 1$ and ignoring the additive constant.

□

To conclude, both functions U and $\overline{U}$ represent $\succ$ over C. So, they are strictly increasing transformations of one another. Letting $\overline{G}$ denote the function of $\overline{U}$ such that $\overline{G}(\overline{U}(c)) = G(U(c))$ for all c, we obtain representation (5). For uniqueness, note that the additive form of $\overline{U}$ is unique up to positive affine transformations, i.e., $\tilde{U} = \rho\overline{U} + \chi$ for $\rho > 0$ and $\chi \in \mathbb{R}$. So,

$$\tilde{U}(c) = \rho\hat{u}(c_0) + \chi + \sum_{t=1}^{\infty} \alpha^t \rho G(\overline{U}({}_t c)) = \rho\hat{u}(c_0) + \chi + \sum_{t=1}^{\infty} \alpha^t \rho G\left(\frac{\tilde{U}({}_t c) - \chi}{\rho}\right).$$

A.3 Proof of Proposition 3

Part (i). Take $\nu', \nu \in \mathcal{U}$. By definition, there exist $c', c \in C$ such that $U(c') = \nu'$ and $U(c) = \nu$. By Lemma 13, we can take $c' = (x, x, \ldots)$ and $c = (y, y, \ldots)$ for some $x, y \in X$. Suppose $u(x) > u(y)$. Then, by Lemma 12, $U(x, \ldots) > U(y, \ldots)$. By representation (5),

$$U(x) - \frac{\alpha}{1-\alpha} G(U(x)) > U(y) - \frac{\alpha}{1-\alpha} G(U(y)).$$

Rearranging, we get that for any $\nu' > \nu$ in $\mathcal{U}$

$$G(\nu') - G(\nu) < \frac{1-\alpha}{\alpha}(\nu' - \nu).$$

Lemma 17. *For any $\varepsilon > 0$, there exists a constant $K \in (\frac{1-\alpha}{2\alpha}, \frac{1-\alpha}{\alpha})$ such that, for all $\nu' > \nu$ in $\mathcal{U}$,*

$$G(\nu') - G(\nu) \leq \max\{K(\nu' - \nu), \varepsilon\} \tag{21}$$

Proof. See Appendix B (Online Appendix).

□

To show that U is H-continuous, consider any $c, \tilde{c} \in C$ and define $c^T = (c_0, c_1, \ldots, c_T, c)$ and $\tilde{c}^T = (c_0, c_1, \ldots, c_T, \tilde{c})$. Using Lemma 17, we will show that for any $\varepsilon > 0$, there exists T such that

$$|U(cT) - U(\tilde{c}T)| < \frac{2\alpha\varepsilon}{1-\alpha}. \tag{22}$$

To do so, let $M = \frac{\alpha}{1-\alpha} 2 \sup_{\nu \in \mathcal{U}} |G(U)|$ and $\delta = (1 + K)\alpha)$. Since $K < (1-\alpha)/\alpha$, we have $\delta < 1$. Let T denote the first time such that $KM\delta^T < \varepsilon$. Note that for all $t < T$, we have $\max\{KM\delta^t, \varepsilon\} = KM\delta^t$.

We first show that for all $t < T$, we have $|U(c^t) - U(\tilde{c}^t)| \leq M\delta^t$. The proof works by induction. For $t = 0$, we have $c_0^t = \tilde{c}_0^t$, so

$$|U(c^0) - U(\tilde{c}^0)| = \sum_{s=1}^{\infty} \alpha^s |G(U({}_s c^0)) - G(U({}_s \tilde{c}^0))| \leq M$$

Suppose the claim holds for $t < T - 1$, we will show it holds for $t + 1$. We have

$$\begin{aligned} |U(c^{t+1}) - U(\tilde{c}^{t+1})| \quad \leq \quad & \alpha |G(U({}_1 c^{t+1})) - G(U({}_1 \tilde{c}^{t+1}))| \\ & + \alpha \sum_{s=1}^{\infty} \alpha^s |G(U({}_{s+1} c^{t+1})) - G(U({}_{s+1} \tilde{c}^{t+1}))|. \end{aligned} \tag{23}$$

By the induction hypothesis, the sum in (23) is bounded above by $M\delta^t$. And because $t < T - 1$, we have $KM\delta^t \geq \varepsilon$. Therefore,

$$|U(c) - U(c')| \leq \alpha KM\delta^t + \alpha M\delta^t \leq M\delta^{t+1},$$

which shows the claim.

Finally, for $t = T$, (23) still applies, but this time the first term is bounded by $\alpha\varepsilon$, because $KM\delta^T < \varepsilon$. This implies that

$$|U(c) - U(c')| \leq \alpha\varepsilon + \alpha M\delta^T \leq \alpha\varepsilon + \alpha\varepsilon/K = \delta\varepsilon/K.$$

Since $\delta < 1$ and $K > (1-\alpha)/2\alpha$, (22) follows.

Part (ii). Let $C(M)$ be the set of consumption streams such that $|u(c_t)| \leq M$ for all t, and

$B(M)$ be the space of bounded real-valued functions with domain $C(M)$. Endowed with the sup norm $\|U\|_\infty = \sup_{c\in C(M)} |U(c)|$, $B(M)$ is a complete metric space. Let $\mathcal{J}$ be the operator on $B(M)$ defined by

$$\mathcal{J}(U)(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)).$$

By construction, $\mathcal{J}(U)$ is bounded over $C(M)$, as u is bounded by M and U is bounded over $C(M)$. Moreover, since G is K-Lipschitz continuous with $K < (1-\alpha)/\alpha$, $\mathcal{J}$ must be a contraction, as is easily checked. So, $\mathcal{J}$ has a unique fixed point; call it U_M. As M increases, the domain of U_M increases. However, for any M, N, uniqueness of the fixed point guarantees that U_M and U_N coincide on the intersection of their domains. Thus, we obtain a unique solution U^* to (5) over $C(B) = \cup_M C(M)$.

Let $\mathcal{H}$ be the set of H-continuous functions. To verify that $U^* \in \mathcal{H}$, it suffices to show that (a) $\mathcal{J}$ maps $\mathcal{H}$ onto itself, and (b) $\mathcal{H}$ is closed under the sup norm. Indeed, this will guarantee that $\mathcal{J}$'s fixed-point belongs to $\mathcal{H}$. To show (a), take any $U \in \mathcal{H}$ and $\varepsilon > 0$. Since $\alpha < 1$ and G is bounded, there is $T > 0$ such that $\frac{\alpha^T 2\bar{G}}{1-\alpha} < \varepsilon/2$, where $\bar{G} = \sup_{\nu\in\mathcal{U}} |G(\nu)|$. Moreover, since $U \in \mathcal{H}$, there exists N such $|U(c) - U(\tilde{c})| < \varepsilon/2$ whenever $c_t = \tilde{c}_t$ for all $t \le N$. For any c and $\tilde{c}$,

$$\begin{aligned} |\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| &\le \left| \sum_{t=1}^{\infty} \alpha^t \left[G(U({}_tc)) - G(U({}_t\tilde{c}))\right] \right| \\ &\le K \sum_{t=1}^{T-1} \alpha^t |U({}_tc) - U({}_t\tilde{c})| + \alpha^T \frac{2\bar{G}}{1-\alpha}. \end{aligned}$$

where K is the Lipschitz constant of G. The first term is less than $\frac{K\alpha}{(1-\alpha)} \max_{t\le T-1} |U({}_tc) - U({}_t\tilde{c})|$. Now suppose that $c_t = \tilde{c}_t$ for all $t \le N' = N + T$. This implies that $({}_tc)_{t'} = ({}_t\tilde{c})_{t'}$ for all $t \le T$ and $t' \le N$, because ${}_tc$ is truncating at most T elements of c, and c and $\tilde{c}$ were identical up to time $T+N$, by construction. By definition of N, we have $|U({}_tc) - U({}_t\tilde{c})| < \varepsilon/2$ for all $t \le T$ and, hence, $|\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| < \varepsilon$. Setting $T(\varepsilon) = N'$ shows that $\mathcal{J}(U)$ satisfies H-continuity. To prove (b), consider a sequence $\{U^m\}$ in $\mathcal{H}$ that converges to some limit U in the sup norm. Now fix $\varepsilon > 0$. There is m such that $\|U^m - U\|_\infty < \varepsilon/3$. Since $U^m \in \mathcal{H}$, there is N such that $|U^m(c) - U^m(\tilde{c})| < \varepsilon/3$ whenever $c_t = \tilde{c}_t$ for all $t \le N$. Thus, for such $c, \tilde{c}$,

$$|U(c) - U(\tilde{c})| \le |U(c) - U^m(c)| + |U^m(c) - U^m(\tilde{c})| + |U^m(\tilde{c}) - U(\tilde{c})| < \varepsilon,$$

which shows that $U \in \mathcal{H}$.

To extend the definition of U^* from $C(B)$ to C, for any $c \in C \setminus C(B)$, consider any sequence $\{c^n\}$ in $C(B)$ such that $c^n_t = c_t$ for all $t \le n$, and let $U^*(c) = \lim_{n\to+\infty} U^*(c^n)$. This limit is well-defined and independent of the chosen sequence. To see this, note that, for any such sequence

$\{c^n\}$ and any $\varepsilon > 0$, H-continuity of U^* implies that there is T such that $|U^*(c) - U^*(\tilde{c})| < \varepsilon$ whenever $c_t = \tilde{c}_t$ for all $t \leq T$. Hence, $|U^*(c^n) - U^*(c^m)| < \varepsilon$ for all $n, m \geq T$, since the consumption levels of c^n and c^m coincide up to $\min\{n, m\}$. So, $\{U^*(c^n)\}$ forms a Cauchy sequence in $\mathbb{R}$ and thus converges. Moreover, the limit is independent of the chosen sequence, as for any $\varepsilon > 0$, $|U^*(c^n) - U^*(\tilde{c}^n)| < \varepsilon$ for n large enough and sequences $\{c^n\}$ and $\{\tilde{c}^n\}$ of the type constructed above.

The limit U thus defined satisfies representation (5). Since U^* is a fixed point of $\mathcal{J}$ on $C(B)$ and c^n belongs to $C(B)$, for each n

$$U^*(c^n) = u(c_0^n) + \sum_{t=1}^{\infty} \alpha^t G(U^*({}_tc^n))$$

The left-hand side converges to $U^*(c)$. Moreover, for each t, $U^*({}_tc^n)$ converges to $U^*({}_tc)$ (which is similarly well defined). Since G is continuous, $G(U^*({}_tc^n))$ converges to $G(U^*({}_tc))$ for each t. Since $\alpha < 1$ and G is bounded, by the dominated convergence theorem, the right-hand side converges to $u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U^*({}_tc))$, which proves that (5) holds for all $c \in C$.

Finally, there is a unique H-continuous extension of U^* from $C(B)$ to C that solves (5). To see this, let U be any H-continuous solution to (5). Since U is a fixed point of $\mathcal{J}$ and the fixed point is unique on $C(B)$, U must coincide with U^* on $C(B)$. Take any $c \in C \setminus C(B)$ and $\varepsilon > 0$. By H-continuity of U and U^*, both $|U(c) - U(\tilde{c})|$ and $|U^*(c) - U^*(\tilde{c})|$ are less than $\varepsilon/2$ for some $\tilde{c} \in C(B)$ equal to c for all t up to a large N. Since U and U^* must be equal at $\tilde{c}$, $|U(c) - U^*(c)| < \varepsilon$. Since ε was arbitrary, $U(c) = U^*(c)$ for all c, establishing uniqueness.

A.4 Proof of Theorem 4

Using Axiom 10 and Theorem 3, we also have

$$(c_0, c_1, {}_2c) \succ (c_0, c_1', {}_2c) \Leftrightarrow (\hat{c}_0, c_1, {}_2c') \succ (\hat{c}_0, c_1', {}_2c') \tag{24}$$

$$(c_0, c_1, {}_2c) \succ (c_0, c_1, {}_2c') \Leftrightarrow (\hat{c}_0, c_1', {}_2c) \succ (\hat{c}_0, c_1', {}_2c') \tag{25}$$

$$(c_0, c_1, {}_2c) \succ (c_0', c_1, {}_2c) \Leftrightarrow (c_0, c_1', {}_2c') \succ (c_0', c_1', {}_2c') \tag{26}$$

$$(c_0, c_1, {}_2c) \succ (c_0, c_1', {}_2c') \Leftrightarrow (\hat{c}_0, c_1, {}_2c) \succ (\hat{c}_0, c_1', {}_2c') \tag{27}$$

By Debreu's (1960) Theorem 3, conditions (24)-(27) and (i)-(ii) in Axiom 10 imply that $\succ$ can be represented by

$$w_0(c_0) + w_1(c_1) + w_2({}_2c),$$

for some continuous and nonconstant functions w_0, w_1, and w_2. By Theorem 3, $\succ$ is also represented by

$$u(c_0) + \alpha G(u(c_1) + g({}_2c)) + \alpha g({}_2c),$$

where $g({}_2c) = \sum_{t=2}^{\infty} \alpha^{t-1} G(U({}_tc))$. It follows that

$$u(c_0) + \alpha G(u(c_1) + g({}_2c)) + \alpha g({}_2c) = \xi \left[w_0(c_0) + w_1(c_1) + w_2({}_2c) \right] + \chi,$$

for some $\xi > 0$ and $\chi \in \mathbb{R}$. This implies that

$$\alpha G(u(c_1) + g({}_2c)) + \alpha g({}_2c) = \xi \left[w_1(c_1) + w_2({}_2c) \right],$$

and therefore G must be affine. Since G must be increasing, without loss of generality let $G(U) = \gamma U$ with $\gamma > 0$. Finally, by Proposition 3, $\gamma < \frac{1-\alpha}{\alpha}$.

A.5 Proof of Proposition 5

By assumption, for all t,

$$U({}_tc) = u(c_t) + \sum_{\tau=t+1}^{\infty} \beta \delta^{\tau - t} u(c_\tau), \tag{28}$$

where $0 < \beta = \frac{\gamma}{1+\gamma} < 1$, $0 < \delta = (1+\gamma)\alpha < 1$, $0 < \alpha < 1$.

For the "if part" see the main text. For the "only if" part, using (28), we get

$$\sum_{t=0}^{\infty} w(t) U({}_tc) = w(0) u(c_0) + \sum_{t=1}^{\infty} u(c_t) \left[w(t) + \beta \delta^t \left(\sum_{\tau=0}^{t-1} \frac{w(\tau)}{\delta^\tau} \right) \right].$$

By assumption, $\sum_{t=0}^{\infty} w(t) U({}_tc) = \sum_{t=0}^{\infty} \delta^t u(c_t)$. So the coefficients of $u(c_t)$ must match for all t. For $t = 0$, $w(0) = 1$. Then, for $t = 1$, $w(1) = (1-\beta)\delta = \alpha$. Now suppose $w(t) = \alpha^t$ for all $t = 0, \ldots, \tau$. Then,

$$w(\tau + 1) = \delta^{\tau+1} - \beta \delta^{\tau+1} \frac{1 - \frac{\alpha^{\tau+1}}{\delta^{\tau+1}}}{1 - \frac{\alpha}{\delta}} = \alpha^{\tau+1}.$$

Hence, by induction, $w(t) = \alpha^t$ for all t.

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B Online Appendix: Omitted Proofs

B.1 Proof of Theorem 1

The proof follows and generalizes that of Diamond (1965), and is based on the following lemmas.

Lemma 18 (Debreu (1954))**.** *Let C be a completely ordered set and $Z = (z_0, z_1, ...)$ be a countable subset of C. If for every $c, c' \in C$ such that $c \prec c'$, there is $z \in Z$ such that $c \precsim z \precsim c'$, then there exists on C a real, order-preserving function, continuous in any natural topology.*[36]

Lemma 19. *For any $c \in C$, there exists $x \in X$ such that $c \sim (c_0, x, x, \ldots)$.*

Proof. Given c, let $D_c = \{(c_0, y, y, \ldots) : y \in X\}$, $A = \{d \in D_c : d \precsim c\}$, and $B = \{d \in D_c : d \succsim c\}$. By Axiom 1, $A \cup B = D_c$; by Axiom 2, A and B are closed; by Axiom 3, A and B are nonempty. Moreover, D_c is connected. Indeed, for any continuous function $\phi : D_c \to \{0, 1\}$, the function $\bar{\phi} : X \to \{0, 1\}$ defined by $\bar{\phi}(x) = \phi(c_0, x, x, \ldots)$ is also continuous. Connectedness of X implies that $\bar{\phi}$ is constant and, hence, that ϕ is constant, showing connectedness of D_c. This implies that $A \cap B \neq \varnothing$.

□

To conclude the proof of Theorem 1, let Z_0 be a countable dense subset of X, which exists since X is separable, and let Z be the subset of C consisting of streams $(x, y, y, \ldots)$ with $x, y \in Z_0$. Lemma 19 implies that Z satisfies the hypothesis of Lemma 18, which yields the result. Indeed, by Lemma 19 there are $x, y \in X$ such that $(c_0, x, x, \ldots) \sim c \prec c' \sim (c'_0, y, y, \ldots)$. Consider the set $E \subset X^2$ consisting of (z, w) such that $(c_0, x, x, \ldots) \prec (z, w, w, \ldots) \prec (c'_0, y, y, \ldots)$. E is nonempty by connectedness of X and open by Axiom 2. Since Z is dense in X^2, E must contain an element of Z.

B.2 Proof of Proposition 1

Recall that by assumption $\succ^t = \succ^0$ for all $t \geq 0$. Suppose that $V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c))$ for all $c \in C$ and V is strictly increasing in $U({}_1c)$. If ${}_1c \sim^1 {}_1c'$, then $U({}_1c) = U({}_1c')$ and, since V is a function, $V(c_0, U({}_1c)) = V(c_0, U({}_1c'))$; hence $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. If ${}_1c \succ^1 {}_1c'$, then $U({}_1c) > U({}_1c')$ and, since V is strictly increasing in its second argument, $V(c_0, U({}_1c)) > V(c_0, U({}_1c'))$; hence $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$.

Suppose ${}_1c \sim^1 {}_1c'$ implies $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. Then, for any $(U({}_1c), U({}_2c), \ldots)$ and $(U({}_1c'), U({}_2c'), \ldots)$ such that $U({}_1c) = U({}_1c')$,

$$V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c'), U({}_2c'), \ldots).$$

[36] A natural topology is one under which Axiom 2 holds for that topology.

So V can depend only on its first two arguments. Suppose ${}_1c \succ^1 {}_1c'$ implies $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$. Then, $U({}_1c) > U({}_1c')$. Moreover, it must be that $V(c_0, U({}_1c)) > V(c_0, U({}_1c'))$; that is, V must be strictly increasing in its second argument.

B.3 Proof of Corollary 1

By Theorem 3, $\succ$ can be represented by

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)).$$

Since $(x, c) \succ (y, c)$, $u(x) = u(y) + \overline{u}$ for some $\overline{u} > 0$. Hence, for any $t > 0$, $U(c^x) - U(c^y)$ equals $\overline{u} - \sum_{s=1}^{t} \alpha^s \Delta G_s$, where ΔG_s is defined recursively as follows: for $s = t$,

$$\Delta G_t = G(U({}_tc^y)) - G(U({}_tc^y) - \overline{u}),$$

otherwise

$$\Delta G_s = G(U_s({}_sc^y)) - G\left(U_s({}_sc^y) - \sum_{k=1}^{t-s} \alpha^k \Delta G_{s+k}\right).$$

By Proposition 3, $\Delta G_t < \frac{1-\alpha}{\alpha}\overline{u}$ and

$$\begin{aligned}\Delta G_{t-1} &= G(U_{t-1}({}_{t-1}c^y)) - G\left(U_{t-1}({}_{t-1}c^y) - \alpha \Delta G_t\right) \\ &< (1-\alpha)\Delta G_t < \frac{(1-\alpha)^2}{\alpha}\overline{u}.\end{aligned}$$

Now, suppose that, for all k such that $s < k \leq t-1$, $\Delta G_k < \frac{(1-\alpha)^2}{\alpha}\overline{u}$. It follows that

$$\begin{aligned}\Delta G_s &< \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s} \alpha^\tau \Delta G_{s+\tau}\right] < \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s-1} \alpha^\tau \frac{(1-\alpha)^2}{\alpha} + \alpha^{t-s}\frac{(1-\alpha)}{\alpha}\right]\overline{u} \\ &= \frac{(1-\alpha)^2}{\alpha}\left[\sum_{\tau=0}^{t-s-2} \alpha^\tau (1-\alpha) + \alpha^{t-s-1}\right]\overline{u} = \frac{(1-\alpha)^2}{\alpha}\overline{u}.\end{aligned}$$

Therefore,

$$\sum_{s=1}^{t} \alpha^s \Delta G_s < \overline{u}\left[\alpha^t \frac{1-\alpha}{\alpha} + \sum_{s=1}^{t-1} \alpha^s \frac{(1-\alpha)^2}{\alpha}\right] = \overline{u}(1-\alpha).$$

We conclude that $U(c^x) - U(c^y) > \alpha\overline{u} > 0$.

B.4 Proof of Corollary 2

By representation (5), U clearly depends on c_0 only through $u_0 = u(c_0)$. This implies that $U({}_1c)$—and hence also $U(c)$ (from (5))—depends on c_1 only through $u_1 = u(c_1)$. By induction, $U(c)$ depends on $(c_0, \ldots, c_t)$ only through $(u_0, \ldots, u_t)$, for each t. There remains to establish the result at infinity: If c and $\tilde{c}$ are two streams such that $u(c_t) = u(\tilde{c}_t)$ for all t, we need to show that $U(c) = U(\tilde{c})$. From the previous step, assume without loss of generality that $c_t = \tilde{c}_t$ for all $t \leq T$, where T is any large, finite constant. Since U is H-continuous, we can choose T so that $|U(c') - U(\tilde{c}')| < \varepsilon$ for all $c', \tilde{c}'$ that coincide up to T. Since c and $\tilde{c}$ satisfy this property, $|U(c) - U(\tilde{c})| < \varepsilon$, and since ε was arbitrary, $U(c) = U(\tilde{c})$. This shows that the sequence $\{u_t = u(c_t)\}_{t=0}^{\infty}$ of period-utility levels entirely determines the value of $U(c)$, proving the result.

B.5 Proof of Proposition 4

Consider representation (5) in Theorem 3. For every $c \in C$, we have sequences $\{u_s\}_{s=0}^{\infty}$ and $\{U_s\}_{s=0}^{\infty}$, where $u_s = u(c_s)$ and $U_s = \hat{U}(u_s, u_{s+1}, \ldots)$. Since u is continuous and X is connected, the range of u is a connected interval $\mathcal{I}_u \subset \mathbb{R}$. Recall that the range of U is also a connected interval $\mathcal{U} \subset \mathbb{R}$. Using the notation,

$$d(t, c) = \frac{\partial U_0 / \partial u_t}{\partial U_0 / \partial u_0}.$$

Note that $\frac{\partial U_s}{\partial u_s} = 1$ for all $s \geq 0$. Since G is differentiable, we have

$$\frac{\partial U_0}{\partial u_t} = \sum_{\tau=0}^{t-1} \alpha^{t-\tau} G'(U_{t-\tau}) \frac{\partial U_{t-\tau}}{\partial u_t}.$$

More generally, for $1 \leq \tau \leq t$,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \sum_{s=0}^{\tau-1} \alpha^{\tau-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t}.$$

So, for $\tau = 1$, $\frac{\partial U_{t-1}}{\partial u_t} = \alpha G'(U_t)$. More generally, for $2 \leq \tau \leq t$,

$$\begin{aligned}
\frac{\partial U_{t-\tau}}{\partial u_t} &= \alpha \sum_{s=0}^{(\tau-1)-1} \alpha^{(\tau-1)-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t} + \alpha G'(U_{t-(\tau-1)}) \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \\
&= \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \alpha (1 + G'(U_{t-(\tau-1)})).
\end{aligned}$$

So,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \alpha^\tau G'(U_t) \prod_{s=1}^{\tau-1}(1+G'(U_{t-s})).$$

Let $\prod_{s=1}^{\tau-1}(1+G'(U_{t-s})) = 1$ if $\tau = 1$. Then,

$$\begin{aligned}\frac{\partial U_0}{\partial u_t} &= \alpha^t G'(U_t) + G'(U_t)\sum_{\tau=1}^{t-1}\alpha^t G'(U_{t-\tau})\prod_{s=1}^{\tau-1}(1+G'(U_{t-s})) \\ &= \alpha^t G'(U_t)\left[1+\sum_{\tau=1}^{t-1}G'(U_{t-\tau})\prod_{s=1}^{\tau-1}(1+G'(U_{t-s}))\right].\end{aligned}$$

B.6 Proof of Corollary 3

For every $c \in C$, consider the sequence $\{U_s\}_{s=0}$ in the proof of Proposition 4. Using representation (5) and Axiom 7(ii) as in Lemma 12, we have that $c \geq_u c'$ implies $U_s \geq U'_s$ for all $s \geq 0$. It is immediate that, if G' is increasing (decreasing), then $d(t,c) \geq (\leq)\ d(t,c')$ for all $t > 0$. On the other hand, suppose G' is not increasing, i.e., there is $U > U'$ in $\mathcal{U}$ such that $G'(U) < G'(U')$—the other case is similar. By definition and Lemma 19, $U = U(c)$ and $U' = U(c')$ for some constant streams c and c'. By Lemma 12, $c \geq_u c'$. However, for all $t > 0$, $d(t,c) < d(t,c')$.

B.7 Proof of Lemma 17

Recall that for any $\nu' > \nu$ in $\mathcal{U}$

$$G(\nu') - G(\nu) < \frac{1-\alpha}{\alpha}(\nu' - \nu).$$

We will show that, for any $\varepsilon > 0$ small enough, there exists a constant $K < \frac{1-\alpha}{\alpha}$ such that

$$G(\nu') - G(\nu) \leq \max\{K(\nu' - \nu), \varepsilon\} \tag{29}$$

for all $\nu' > \nu$ in $\mathcal{U}$.

Case (i): Suppose first that $\mathcal{U}$ is bounded and let $\overline{\mathcal{U}} = cl(\mathcal{U})$. If necessary, extend G to $\overline{\mathcal{U}}$ by continuity. Since $\overline{\mathcal{U}}$ is compact and G is continuous, it is also uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $|\nu - \nu'| < \eta(\varepsilon)$ implies $|G(\nu) - G(\nu')| < \varepsilon$. Let $\Delta(\varepsilon) = \{(\nu,\nu') \in \overline{\mathcal{U}}^2 \mid \nu \geq \nu' + \eta(\varepsilon)\}$. The function $F(\nu,\nu') = \frac{G(\nu)-G(\nu')}{\nu-\nu'}$ is continuous and strictly less[37] than $\frac{1-\alpha}{\alpha}$ on the compact set $\Delta(\varepsilon)$ and thus has a strictly positive upper bound

[37]This is true by assumption if ν and ν' belong to $\mathcal{U}$, and it is easy to show that it is still true if either ν or ν' belongs to $\overline{\mathcal{U}} \setminus \mathcal{U}$. For example, if ν' is the infimum of $\mathcal{U}$, one can take any point $\tilde{\nu} \in (\nu',\nu)$. By assumption $G(\nu) - G(\tilde{\nu}) < (1-\alpha)/\alpha(\nu - \tilde{\nu})$ and, by continuity of G, $G(\tilde{\nu}) - G(\nu') \leq (1-\alpha)/\alpha(\tilde{\nu} - \nu')$. Combining these inequalities yields the result, as is easily seen. (One way of showing this is to use the

$K < \frac{1-\alpha}{\alpha}$. By construction, (29) holds for any $(\nu, \nu') \in \Delta(\varepsilon)$ and any $(\nu, \nu') \in \overline{\mathcal{U}}^2 \setminus \Delta(\varepsilon)$.

Case (ii): Suppose that $\mathcal{U}$ is unbounded both above and below—the intermediate cases follow by combining the two cases shown here. Let $\underline{G} = \inf_{\nu \in \mathcal{U}} G(\nu)$ and $\overline{G} = \sup_{\nu \in \mathcal{U}} G(\nu)$, which are finite and distinct because G is bounded and strictly increasing. Fix any $\varepsilon < \overline{G} - \underline{G}$. Let $\underline{\nu}(\varepsilon) = G^{-1}(\underline{G} + \varepsilon)$ and $\overline{\nu}(\varepsilon) = G^{-1}(\overline{G} - \varepsilon)$. If either $\nu \leq \underline{\nu}(\varepsilon)$ and $\nu' \leq \underline{\nu}(\varepsilon)$, or $\nu \geq \overline{\nu}(\varepsilon)$ and $\nu' \geq \overline{\nu}(\varepsilon)$, then (29) holds by construction. Now take any $\overline{\nu}, \underline{\nu} \in \mathcal{U}$ with $\overline{\nu} > \overline{\nu}(\varepsilon) + 2(\frac{\alpha\varepsilon}{1-\alpha} + 1)$ and $\underline{\nu} < \underline{\nu}(\varepsilon) - 2(\frac{\alpha\varepsilon}{1-\alpha} + 1)$. On the compact set $[\underline{\nu}, \overline{\nu}]$, the continuous function G is uniformly continuous, so there exists $\eta > 0$ and $\eta(\varepsilon) = \min\{\eta, \frac{1}{2}(\overline{v} - \overline{\nu}(\varepsilon)), \frac{1}{2}(\underline{\nu}(\varepsilon) - \underline{\nu})\}$ such that $|\nu - \nu'| < \eta(\varepsilon)$ implies $|G(\nu) - G(\nu')| < \varepsilon$. Let $\Delta'(\varepsilon) = \{(\nu, \nu') \in [\underline{\nu}, \overline{\nu}]^2 \mid \nu \geq \nu' + \eta(\varepsilon)\}$. By the same argument as before, the function $F(\nu, \nu') = \frac{G(\nu) - G(\nu')}{\nu - \nu'}$ has a strictly positive upper bound $K_1 < \frac{1-\alpha}{\alpha}$ on the set $\Delta'(\varepsilon)$.

Define $\overline{\nu}_m = \frac{1}{2}(\overline{\nu} + \overline{\nu}(\varepsilon))$ and $\underline{\nu}_m = \frac{1}{2}(\underline{v} + \underline{\nu}(\varepsilon))$. The only difficulty is to show the claim when $\nu' < \overline{\nu}(\varepsilon) \leq \overline{\nu} < \nu$ or $\nu' < \underline{v} \leq \underline{\nu}(\varepsilon) < \nu$. We focus on the first case. If $\nu' < \overline{v}(\varepsilon)$, by construction $\overline{\nu}_m - \nu' \geq \eta(\varepsilon)$ and hence

$$\frac{G(\overline{\nu}_m) - G(\nu')}{\overline{\nu}_m - \nu'} < K_1. \tag{30}$$

Now note that

$$\nu - \overline{\nu}_m > \overline{\nu} - \overline{\nu}_m = \frac{1}{2}(\overline{\nu} - \overline{\nu}(\varepsilon)) > \frac{\alpha\varepsilon}{1-\alpha} + 1.$$

Hence, there exists a strictly positive $K_2 < \frac{1-\alpha}{\alpha}$ such that, for all $\nu > \overline{\nu}$, we have $\nu - \overline{\nu}_m > \varepsilon/K_2$. Since $\nu > \overline{\nu}(\varepsilon)$ and $\overline{\nu}_m > \overline{\nu}(\varepsilon)$, it follows that

$$\frac{G(\nu) - G(\overline{\nu}_m)}{\nu - \overline{\nu}_m} \leq \frac{\varepsilon}{\nu - \overline{\nu}_m} < K_2. \tag{31}$$

For any strictly positive a, b, c, d, $(a + c)/(b + d) \leq \max\{a/b, c/d\}$. Combining this inequality to (30) and (31), we conclude that

$$\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_2\}.$$

By a similar argument, for all $\nu' < \underline{v} \leq \underline{\nu}(\varepsilon) < \nu$,

$$\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_3\}$$

for some strictly positive $K_3 < \frac{1-\alpha}{\alpha}$. Letting $K = \max\{K_1, K_2, K_3\}$ then proves the claim of the lemma.

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fact that $a/b < c/d \Rightarrow (a + b)/(c + d) < c/d$ for a, b, c, d strictly positive—see the argument at the end of this proof.)