

Optimal Decumulation of Assets in General Equilibrium*

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Abstract

As the Baby Boom passes into retirement, policymakers and economists will have to shift their focus from figuring out how to encourage aging workers to save more to figuring out how the elderly should optimally dispose of their savings. Conventional wisdom among economists says it is best for a household with no bequest motive to invest its assets in an annuity. However, very few people actually follow this advice, and recent work has shown that the economy can be better off in general equilibrium if households do not annuitize their savings. I consider the optimal decumulation of assets in a general equilibrium model under three regimes. In one regime, households are fully rational but only have access to realistic annuities that pay a constant stream of income until death, as opposed to perfect Yaari (1965) annuities for which the return increases with mortality risk. In the second regime, households have the same choices but follow the restricted optimal irrational behavior paradigm with Keynesian consumption functions. In the third regime, households are fully rational but are forced to invest their savings in some form of annuity at retirement. Though in partial equilibrium, households are better off participating in longevity annuities that assimilate assets upon death and therefore deliver higher returns than bonds, it is suboptimal for households to have access to longevity annuities unless the discount rate is very low or intertemporal elasticity is very high. Instead, households should generally be encouraged to participate in “bond annuities”, which offer the same internal rate of return as bonds while paying a constant stream of income since these conserve more wealth than a household would if it managed its portfolio with plain bonds. Upon death, ownership of the assets in a bond annuity is retained by the household’s estate, which recirculates this wealth to a greater part of the population than would a longevity annuity.

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For my entire four-decade lifespan, one of the prime concerns of lifecycle economists has been preparing the Baby Boomer generation for their eventual retirement, fearing that Social Security would not be up to the task of financing their consumption needs. Now that eventuality has come to pass. The Baby Boomers have largely accumulated whatever savings they are going to have, and the leading question will become how they should best decumulate this saving.

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Since Yaari (1965) first studied this question, most economists have advised households to invest their savings in an annuity that will provide a stream of income until the household dies. In this initial paper, Yaari focused exclusively on perfect annuities: at every instant investors with similar characteristics pool their funds, which are then divvied up by those members of the pool that survive to the next instant. This provides perfect insurance against mortality risk, and a household without a bequest motive should put as much as is allowed in this idealized annuity.¹ In the real world, most households do not have access to perfect annuities. Instead, they can invest with an annuity company that pays out a constant stream of income till the household dies. The return of such a real annuity will be constant, in contrast to a perfect annuity, which pays an age-dependent return that increases with the hazard rate of dying. However, the pooling of funds across a slowly attriting group of households still garners a higher return than unpooled investments, so the theoretical result that households should participate in annuities is extremely robust (Davidoff, Brown, and Diamond (2005)).

Empirically, though, most households do not participate in annuities. Johnson, Burman, and Kobes (2004) report that less than 1% of US retirement wealth has been invested in annuities, and Pashchenko (2011) finds that less than 6% of households over 65 participate in any form of marketed annuity.² These participation rates are likely further inflated because surveys do not distinguish between true annuities that provide longevity insurance and what we will call bond annuities that merely pay out a constant income stream without assimilating the underlying assets. Households are forced to participate in Social Security, which is a non-marketed annuity, but it pays a return less than the normal return on investments.³ This data has led many economists to ask what can be done to encourage more people to invest in annuities. The present paper asks whether annuitization actually improves welfare in general equilibrium and, if so, what kinds of annuities improve welfare the most.

Yaari's (1965) result was obtained in partial equilibrium, treating factor prices and, quite importantly, the prospect of inheriting a bequest as exogenous variables. Both Feigenbaum, Gahramanov, and Tang (FGT) (2013b) and Heijdra, Mierau, and Reijnders (HMR) (2014) have demonstrated that Yaari's result can be reversed in general equilibrium, and this is the typical outcome for normal parameterizations. Rational households ought to annuitize because of the higher return, but in general equilibrium this has the detrimental side effect that, if everyone annuitizes, no one will leave an estate for the next generation, which implies no one can inherit a bequest.⁴

Rather than consider what happens with perfect, Yaari (1965) annuities, here we focus on more realistic annuities. At retirement, households can irreversibly invest in an annuity that pays out a constant stream of consumption until death. We consider two types of such annuities. Longevity annuities pool the funds of all the investors that retire at time t , and these funds remain in that pool as the members die until the funds are exhausted. Consequently, longevity annuities earn a higher internal rate of return than other investments, although this return is constant, unlike the return on a perfect annuity.

Households can alternatively invest in bond annuities. Bond annuities involve no risk pooling and do not employ any investment technology that individual households lack. Investors in bond annuities simply give their savings over to the annuity company, which invests these funds in regular bonds and pays the household a constant income stream. The funds remain the property of the investor, and the household's heirs inherit these funds when the household dies.⁵ The bond annuity pays the same internal rate of return

¹A household without a bequest motive should allocate a portion of its wealth to its heirs and a portion to its own consumption. The latter portion should be fully annuitized.

²For an extensive review of this annuities puzzle, see Feigenbaum, Gahramanov, and Tang (2013b).

³Moreover, Guo, Caliendo, and Hosseini (2014) have shown that Social Security does not provide any insurance against mortality risk since it pools funds across all cohorts instead of creating a separate pool for each cohort like a longevity annuity would.

⁴Note that all bequests are assumed to be accidental here.

⁵These investment instruments do exist in reality. For example, TIAA-CREF offers three income options from an annuity contract owned by a one-person household. A One-Life Annuity without a guaranteed period is a longevity annuity. A

as bonds, but it is illiquid so rational households would have no need to invest in the bond annuity.

On the other hand, rational households will always invest a positive amount in the longevity annuity because of its higher internal rate of return. If we prohibit households from borrowing after retirement, the portfolio allocation problem of determining how much to invest in the longevity annuity is complicated since the household will only desire a constant consumption stream in the knife-edge case when the interest rate equals the household's discount rate. Results for this model regime are pending.

A simpler model results if we require households to invest all of their funds in one or the other annuity upon retirement, and it is actually quite straightforward to compare what happens under four policy environments. In the Full Annuitization case, I impose this restriction. In the Bond Annuity case, I force households to invest specifically in the bond annuity. In the No Annuitization case, I deny households access to annuities. Finally, for comparison, I also consider the Perfect Annuitization case where households have access to Yaari's idealized annuities.

Not surprisingly, I replicate the findings of FGT (2013b) and HMR (2014) that No Annuitization yields higher utility than Perfect Annuitization except when the elasticity of intertemporal elasticity is extremely high. Since the longevity annuities introduced here pay a lower return on average than perfect annuities, Full Annuitization generally earns a slightly lower utility than Perfect Annuitization. Remarkably, however, when the elasticity of intertemporal substitution is less than 2, it is the Bond Annuity case that gives the highest utility.⁶

While FGT (2013b) and HMR (2014) have indicated that the intuition most economists have about why it is good to annuitize is invalid, this intuition does explain why bond annuities are optimal. The supposed benefits of longevity annuities are twofold: they provide a smoother consumption stream, and they insure against mortality risk. Bond annuities only provide the smoother consumption stream, and this turns out to be the more important benefit. In general equilibrium, the means by which longevity annuities insure against mortality risk is ultimately more detrimental than beneficial. Social Security is another annuity that can help to smooth consumption without insuring against mortality risk. It decreases utility because it reduces the incentives for saving and because it is financed by distortionary taxes (Conesa and Garriga (2008)), but those drawbacks do not apply to bond annuities.

As we build new institutions to help the Baby Boomers decumulate their savings, what matters most is helping them to smooth their consumption stream, which can be done by bond annuities, rather than insuring them against mortality risk. Basu et al (2014) are an example of a paper looking to examine the optimal default choice that employers should pick for a defined-contribution plan, but again they confine their attention to partial equilibrium.

The paper proceeds as follows. Section 1 introduces the basic model that will be considered throughout the paper. Section 2 considers what happens if rational households are forced to invest their retirement savings in some form of annuity. Section 3 computes the restricted optimal irrational behavior with Keynesian consumption functions. Section 4, which is still incomplete, computes the optimal portfolio allocation in general equilibrium for fully rational households that face no additional restrictions beyond the assumption, common to all three regimes, that households cannot borrow after retirement.

1 The General Model

Fixed-Period Annuity is a bond annuity. A One-Life Annuity with a guaranteed period is a hybrid of these two instruments, behaving similar to a bond annuity if the annuitant dies before the end of the guaranteed period and like a longevity annuity thereafter. Only a household with an explicit bequest motive would have an incentive to purchase an annuity with a guaranteed period, so we do not consider them in the present model.

⁶We also consider what happens under the restricted optimal irrational behavior paradigm (Feigenbaum, Gahramanov, and Tang (2013a, b)). If households are restricted to follow Keynesian consumption functions, it will be natural for them to invest in one or the other type of annuity. In this regime we find that, unless households are extremely patient, it is optimal to invest only in the bond annuity.

We will consider three variations of the following discrete-time model. A household lives to a maximum age of T . The probability of surviving to age $t = 0, \dots, T$ is Q_t . Households maximize

$$U = \sum_{t=0}^T Q_t \beta^t u(c_t) \quad (1)$$

where $\beta > 0$ is the discount factor, c_t is consumption at age t , and $u(c)$ is a CRRA utility function with elasticity of intertemporal substitution $\gamma^{-1} > 0$.

At age $T_a \in (0, T - 1)$, the household finishes work and has the option to invest A^l in a longevity annuity or A^b in a bond annuity. The former pays a constant stream $\pi^l A^l$ while the latter pays a constant stream $\pi^b A^b$. In both cases the stream ends when the household dies, but the assets invested in the bond annuity will then be distributed to the household's heirs while the assets invested in the longevity annuity are redistributed to surviving investors. The household can also borrow and lend at the gross risk-free rate R before retirement and can lend at that rate after retirement. During the working life, the household has the endowment e_t of productivity units at age t , which it sells at the real wage w . The household also receives an accidental bequest B .

For $t \in [0, T_a)$, the household's budget constraint is

$$c_t + b_{t+1} = we_t + Rb_t + B. \quad (2)$$

At T_a , the budget constraint is

$$c_{T_a} + b_{T_r} + A_l + A_b = we_{T_a} + Rb_{T_a} + B \quad (3)$$

where

$$A_l, A_b, b_{T_r} \geq 0. \quad (4)$$

Let $T_r = T_a + 1$ be the last period of work. For $t \in [T_r, T]$, the household's budget constraint is

$$c_t + b_{t+1} = Rb_t + \pi_l A_l + \pi_b A_b + B, \quad (5)$$

and the household must satisfy the borrowing constraint

$$b_{t+1} \geq 0. \quad (6)$$

Bond holdings must also satisfy the boundary conditions

$$b_0 = b_{T+1} = 0. \quad (7)$$

We assume the annuities industry is perfectly competitive. The payout rates π_b and π_l are determined so annuities firms make no profit.

First consider the longevity annuity. Assume that everyone who pays in to the annuity company contributes A_l . Let a_t^l be the assets associated with the account of households of age t . Then the assets must satisfy the boundary conditions

$$a_{T_a+1}^l = A_l Q_{T_a} \quad (8)$$

$$a_{T+1}^l = 0. \quad (9)$$

For $t = T_a + 1, \dots, T$, the equation of motion for a_{t+1}^l is

$$\pi_l A_l Q_t + a_{t+1}^l = R a_t^l. \quad (10)$$

Since the annuity company has the same investment technologies as everyone else, it can only invest the assets it manages as capital. The right-hand side of (10) shows the assets available at the beginning of period t , which will be allocated between the payout to currently surviving investors and the assets to continue next period.

Define

$$Q_{s|t} = \frac{Q_s}{Q_t} \quad (11)$$

for $s \geq t$ to be the probability of surviving to s conditional on surviving to t . Then we show in Appendix A.1 that payout rate for the longevity annuity can be written

$$\pi_l = \frac{1}{\sum_{s=T_a+1}^T \frac{Q_{s|T_a}}{R^{s-T_a}}}, \quad (12)$$

and the annuity company will have assets

$$a_t^l = R^{t-T_a-1} A_l Q_{T_a} \left[1 - \frac{\sum_{s=T_a+1}^{t-1} R^{-s} Q_s}{\sum_{s=T_a+1}^T R^{-s} Q_s} \right] \quad (13)$$

belonging to investors of age t .

Now consider the bond annuity. For the longevity annuity, the annuity company could combine the assets of all households of age t since the assets effectively belong to the company and not the households. For the bond annuity, the annuity company has to keep the assets of each household separate since these assets still belong to the household. Let a_t^b be the assets managed by the bond annuity company that belong to the household as of age t . If the household invests A_b , these will satisfy the equation

$$a_{t+1}^b = R a_t^b - \pi_b A_b$$

with the boundary conditions $a_{T_a+1}^b = A_b$ and $a_{T+1}^b = 0$. In Appendix A.2, we show the payout on the bond annuity is

$$\pi_b = \frac{r}{1 - R^{-(T-T_a)}}, \quad (14)$$

which is less than the payout on the longevity annuity:

$$\begin{aligned} \pi &= \frac{1}{\sum_{s=T_a+1}^T \frac{Q_{s|T_a}}{R^{s-T_a}}} > \frac{1}{\sum_{s=T_a+1}^T \frac{1}{R^{s-T_a}}} = \frac{1}{\sum_{s=1}^{T-T_a} \frac{1}{R^s}} \\ &= \frac{1}{\frac{1}{R} \frac{1 - R^{T-T_a}}{1 - R} - 1} = \frac{R \left(\frac{1}{R} - 1 \right)}{\frac{1}{R^{T-T_a}} - 1} = \frac{1 - R}{\frac{1}{R^{T-T_a}} - 1} = \frac{r}{1 - R^{-(T-T_a)}} = \pi_b. \end{aligned}$$

A bond annuity will be financed by assets

$$\begin{aligned} a_t^b &= \left[R^{t-T_r} \left(1 - \frac{\pi_b}{r} \right) + \frac{\pi_b}{r} \right] A_b \\ a_t^b &= \left[-R^{t-T_r} \left(\frac{R^{-(T-T_a)}}{1 - R^{-(T-T_a)}} \right) + \frac{1}{1 - R^{-(T-T_a)}} \right] A_b \\ a_t^b &= \frac{1 - R^{-(T+1-t)}}{1 - R^{-(T-T_a)}} A_b \end{aligned} \quad (15)$$

at age t .

Let

$$K_a^l = \sum_{t=T_a}^T a_{t+1}^l \quad (16)$$

be the capital coming from longevity annuity investments,

$$K_a^b = \sum_{t=T_a}^T Q_t a_{t+1}^b \quad (17)$$

the capital coming from bond annuity and investments, and

$$K_b = \sum_{t=0}^T Q_t b_{t+1} \quad (18)$$

the capital coming from regular bond investments.

The capital stock is

$$K = K_a + K_b. \quad (19)$$

The labor supply is

$$N = \sum_{t=0}^T Q_t e_t. \quad (20)$$

In equilibrium, the bequest balance equation

$$BP = R \sum_{t=0}^T (Q_t - Q_{t+1}) b_{t+1} + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) a_{t+1}^b, \quad (21)$$

where

$$P = \sum_{t=0}^T Q_t \quad (22)$$

is the total population. Finally, let us assume that

$$Y = F(K, N) \quad (23)$$

where F has constant returns to scale. The factor prices are

$$r = r(K) = F_K(K, N) - \delta \quad (24)$$

$$R = R(K) = 1 + r(K) \quad (25)$$

$$w = w(K) = F_N(K, N), \quad (26)$$

so

$$Y = wN + (r + \delta)K.$$

The working budget constraint can be added up from 0 to $t \in [0, T_a)$ to obtain

$$\sum_{s=0}^t \frac{c_s + b_{s+1}}{R^s} = \sum_{s=0}^t \frac{we_s + Rb_s + B}{R^s}$$

$$\frac{b_{t+1}}{R^t} + \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s} = \sum_{s=0}^t \frac{we_s + B - c_s}{R^s} + \sum_{s=1}^t \frac{b_s}{R^{s-1}} + Rb_0$$

Since $b_0 = 0$,

$$\frac{b_{t+1}}{R^t} + \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s} = \sum_{s=0}^t \frac{we_s + B - c_s}{R^s} + \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s}$$

For $t \in [0, T_a)$,

$$b_{t+1} = R^t \sum_{s=0}^t \frac{we_s + B - c_s}{R^s} \quad (27)$$

Thus

$$b_{T_a} = R^{T_a-1} \sum_{s=0}^{T_a-1} \frac{we_s + B - c_s}{R^s}. \quad (28)$$

The present value of consumption for $t = 0, \dots, T_a - 1$ is

$$\sum_{t=0}^{T_a-1} \frac{c_t}{R^t} = \sum_{t=0}^{T_a-1} \frac{we_t + B}{R^t} - \frac{b_{T_a}}{R^{T_a-1}}. \quad (29)$$

Accounting for (3),

$$\begin{aligned} \sum_{t=0}^{T_a-1} \frac{c_t}{R^t} &= \sum_{t=0}^{T_a-1} \frac{we_t + B}{R^t} - \frac{c_{T_a} + b_{T_r} + A_l + A_b - we_{T_a} - B}{R^{T_a}} \\ \sum_{t=0}^{T_a} \frac{c_t}{R^t} &= \sum_{t=0}^{T_a} \frac{we_t + B}{R^t} - \frac{b_{T_r} + A_l + A_b}{R^{T_a}} \end{aligned} \quad (30)$$

Summing the budget constraint for $s = T_r, \dots, t \leq T$ we obtain

$$\begin{aligned} \sum_{s=T_r}^t \frac{c_s + b_{s+1}}{R^s} &= \sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B + Rb_s}{R^s} \\ \sum_{s=T_r}^t \frac{c_s}{R^s} + \sum_{s=T_r}^t \frac{b_{s+1}}{R^s} &= \sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B}{R^s} + \sum_{s=T_r}^t \frac{b_s}{R^{s-1}} \\ \sum_{s=T_r}^t \frac{c_s}{R^s} + \sum_{s=T_r}^{t-1} \frac{b_{s+1}}{R^s} + \frac{b_{t+1}}{R^t} &= \sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B}{R^s} + \frac{Rb_{T_r}}{R^{T_r}} + \sum_{s=T_r+1}^t \frac{b_s}{R^{s-1}} \\ \sum_{s=T_r}^t \frac{c_s}{R^s} + \sum_{s=T_r}^{t-1} \frac{b_{s+1}}{R^s} + \frac{b_{t+1}}{R^t} &= \sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B}{R^s} + \frac{Rb_{T_r}}{R^{T_r}} + \sum_{s=T_r}^{t-1} \frac{b_{s+1}}{R^s} \\ \sum_{s=T_r}^t \frac{c_s}{R^s} + \frac{b_{t+1}}{R^t} &= \sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B}{R^s} + \frac{b_{T_r}}{R^{T_a}} \end{aligned} \quad (31)$$

$$b_{t+1} = R^t \left[\sum_{s=T_r}^t \frac{\pi_l A_l + \pi_b A_b + B - c_s}{R^s} + \frac{b_{T_r}}{R^{T_a}} \right] \quad (32)$$

Applying (31) for $t = T$,

$$\sum_{t=T_r}^T \frac{c_t}{R^t} = \sum_{t=T_r}^T \frac{\pi_l A_l + \pi_b A_b + B}{R^t} + \frac{b_{T_r}}{R^{T_a}}. \quad (33)$$

Adding (30) and (33) we obtain the lifetime budget constraint,

$$\sum_{t=0}^T \frac{c_t}{R^t} = \sum_{t=0}^{T_a} \frac{we_t}{R^t} + \sum_{t=0}^T \frac{B}{R^t} + \sum_{t=T_r}^T \frac{\pi_l A_l + \pi_b A_b}{R^t} - \frac{A_l - A_b}{R^{T_a}}. \quad (34)$$

$$\sum_{t=0}^{t_b} \frac{1}{R^t} = \frac{1}{R^{t_a}} \sum_{t=0}^{t_b-t_a} \frac{1}{R^t} = \frac{1}{R^{t_a}} \frac{1 - \frac{1}{R^{t_b-t_a+1}}}{\frac{1}{R} - 1} = \frac{R}{R^{t_a}} \frac{1 - \frac{1}{R^{t_b-t_a+1}}}{r} = \frac{1 - \frac{1}{R^{t_b-t_a+1}}}{R^{t_a}} \frac{R}{r} \quad (35)$$

$$\sum_{t=0}^T \frac{c_t}{R^t} = \sum_{t=0}^{T_a} \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \frac{1 - \frac{1}{R^{T-T_r+1}}}{R^{T_r}} \frac{R}{r} (\pi_l A_l + \pi_b A_b) - \frac{A_l + A_b}{R^{T_a}}$$

$$\sum_{t=0}^T \frac{c_t}{R^t} = \sum_{t=0}^{T_a} \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_b - 1 \right] \frac{A_b}{R^{T_a}},$$

which simplifies to

$$\sum_{t=0}^T \frac{c_t}{R^t} = \sum_{t=0}^{T_a} \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}}$$

because of (14). Investing in the bond annuity does nothing to change wealth since it is financed by investing in the same bonds that the household has available to it:

Thus lifetime wealth is

$$W = \sum_{t=0}^T \frac{c_t}{R^t} = \sum_{t=0}^{T_a} \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_b - 1 \right] \frac{A_b}{R^{T_a}} \quad (36)$$

Investing in the longevity annuity increases wealth:

$$\begin{aligned} \frac{\partial W}{\partial A_l} &= \frac{1}{R^{T_a}} \left[\sum_{t=T_r}^T \frac{\pi_l}{R^{t-T_a}} - 1 \right] \\ &= \frac{1}{R^{T_a}} \left[\frac{\sum_{t=T_r}^T R^{-(t-T_a)}}{\sum_{t=T_r}^T R^{-(t-T_a)} Q_{t|T_a}} - 1 \right] \end{aligned}$$

Since $Q_{t|T_a} < 1$ for all $t \in [T_r, T]$,

$$\frac{\sum_{t=T_r}^T R^{-(t-T_a)}}{\sum_{t=T_r}^T R^{-(t-T_a)} Q_{t|T_a}} > 1.$$

Thus $\frac{\partial W}{\partial A_l} > 0$. The internal rate of return of this flat annuity is higher than the rate of return on bonds, so, as we will see in Section 4, a rational household will invest as much as is possible in the longevity annuity until the borrowing constraint binds. However, this strategy does not maximize utility in general equilibrium as we see in the following two sections.

2 Restricted Rational Equilibrium

There is much discussion, although not yet much action, by economists and legislators regarding bills that would encourage pensions and 401(k)s to roll over into annuities upon retirement. To see the effect of applying such policies on a large scale, let us suppose that households are rational but are required, presumably by the government, to put any savings either into longevity annuities or bond annuities at age T_a .

The Lagrangian for the household's problem is

$$L_{rr} = \sum_{t=0}^T Q_t \beta^t u(c_t) + \sum_{t=0}^{T_a-1} \lambda_t [we_t + Rb_t + B - c_t - b_{t+1}] + \lambda_{T_a} [we_{T_a} + Rb_{T_a} + B - c_{T_a} - A_b - A_l] \\ + \sum_{t=T_r}^T \lambda_t [\pi_l A_l + \pi_b A_b + B - c_t] + \mu_b A_b + \mu_l A_l$$

$$\frac{\partial L_{rr}}{\partial c_t} = Q_t \beta^t u'(c_t) - \lambda_t = 0$$

For $t = 0, \dots, T_a - 1$,

$$\frac{\partial L_{rr}}{\partial b_{t+1}} = -\lambda_t + R\lambda_{t+1} = 0$$

$$\frac{\partial L_{rr}}{\partial A_b} = -\lambda_{T_a} + \pi_b \sum_{t=T_r}^T \lambda_t + \mu_b = 0$$

$$\frac{\partial L_{rr}}{\partial A_l} = -\lambda_{T_a} + \pi_l \sum_{t=T_r}^T \lambda_t + \mu_l = 0$$

Thus

$$\lambda_{T_a} = \pi_l \sum_{t=T_r}^T \lambda_t + \mu_l = \pi_b \sum_{t=T_r}^T \lambda_t + \mu_b$$

Since $\pi_l > \pi_b$ and $\lambda_t > 0$ for all t , we must have $\mu_b > \mu_l \geq 0$. Thus $\mu_b > 0$ and $A_b = 0$. For all t ,

$$\lambda_t = Q_t \beta^t u'(c_t).$$

For $t = 0, \dots, T_a$,

$$\lambda_t = R^{T_a-t} \lambda_{T_a} \\ R^{T_a-t} \lambda_{T_a} = R(R^{T_a-t-1} \lambda_{T_a}) = R^{T_a-t} \lambda_{T_a}$$

For $t = T_r, \dots, T$,

$$\lambda_t = Q_t \beta^t u'(\pi_l A_l + B) \\ \lambda_{T_a} = \pi_l \sum_{t=T_r}^T Q_t \beta^t u'(\pi_l A_l + B) \\ \lambda_{T_a} = \pi_l u'(\pi_l A_l + B) \sum_{t=T_r}^T Q_t \beta^t$$

For $t = 0, \dots, T_a$,

$$\begin{aligned}\lambda_t &= R^{T_a-t} \pi_l u'(\pi_l A_l + B) \sum_{s=T_r}^T Q_s \beta^s \\ Q_t \beta^t u'(c_t) &= R^{T_a-t} \pi_l u'(\pi_l A_l + B) \sum_{s=T_r}^T Q_s \beta^s \\ c_t &= (u')^{-1} \left(\frac{R^{T_a}}{Q_t \beta^t R^t} \pi_l u'(\pi_l A_l + B) \sum_{s=T_r}^T Q_s \beta^s \right).\end{aligned}$$

If we prohibit households from investing in the longevity annuity, forcing them instead to invest in the bond annuity, we will have

$$\lambda_{T_a} = \pi_b \sum_{t=T_r}^T Q_t \beta^t u'(\pi_b A_b + B)$$

and

$$c_t = (u')^{-1} \left(\frac{R^{T_a}}{Q_t \beta^t R^t} \pi_b u'(\pi_b A_b + B) \sum_{s=T_r}^T Q_s \beta^s \right).$$

The internal rate of return on the bond annuity solves

$$\begin{aligned}1 &= \sum_{t=T_r}^T \frac{\pi_b}{(1+r)^{t-T_a}} = \pi_b \sum_{s=1}^{T-T_a} \frac{1}{(1+r)^s} = \frac{\pi_b}{1+r} \frac{\frac{1}{(1+r)^{T-T_a}} - 1}{\frac{1}{1+r} - 1} \\ &= \pi_b \frac{\frac{1}{(1+r)^{T-T_a}} - 1}{1 - (1+r)} = \frac{1 - \frac{1}{(1+r)^{T-T_a}}}{r} \pi_b = \frac{1 - \frac{1}{(1+r)^{T-T_a}}}{r} \frac{r}{1 - R^{-(T-T_a)}} = 1,\end{aligned}$$

so the internal rate of return is r , as it should be.

With CRRA utility,

$$\begin{aligned}c_{T_a}^{-\gamma} &= \frac{1}{Q_{T_a} \beta^{T_a}} \pi_l (\pi_l A_l + B)^{-\gamma} \sum_{s=T_r}^T Q_s \beta^s \\ c_{T_a}^{-\gamma} &= \pi_l (\pi_l A_l + B)^{-\gamma} \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \\ c_{T_a} &= \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} (\pi_l A_l + B)\end{aligned}\tag{37}$$

For $t = 0, \dots, T_a$,

$$\begin{aligned}c_t &= \left(\frac{Q_{T_a}}{Q_t \beta^{t-T_a} R^{t-T_a}} \pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} (\pi_l A_l + B) \\ c_t &= \left(Q_{T_a|t} \beta^{T_a-t} R^{T_a-t} \right)^{-\frac{1}{\gamma}} c_{T_a} \\ c_t + b_{t+1} &= w e_t + R b_t + B \quad t \in [0, T_a) \\ c_{T_a} + b_{T_r} + A_l &= w e_{T_a} + R b_{T_a} + B\end{aligned}\tag{38}$$

$$\begin{aligned}
\sum_{t=0}^{T_a-1} R^{T_a-t} (c_t + b_{t+1}) + c_{T_a} + A_l &= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + Rb_t + B) \\
\sum_{t=0}^{T_a} R^{T_a-t} c_t + A_l &= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + Rb_t + B) - \sum_{t=0}^{T_a-1} R^{T_a-t} b_{t+1} \\
\sum_{t=0}^{T_a} R^{T_a-t} c_t + A_l &= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + Rb_t + B) - \sum_{t=1}^{T_a} R^{T_a-t+1} b_t \\
\sum_{t=0}^{T_a} R^{T_a-t} c_t + A_l &= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + Rb_t + B) - \sum_{t=0}^{T_a} R^{T_a-t+1} b_t
\end{aligned}$$

Under full annuitization, the annuity will be

$$A_l = \sum_{t=0}^{T_a} R^{T_a-t} (we_t + B - c_t) \quad (39)$$

$$\begin{aligned}
A_l &= \sum_{t=0}^{T_a} R^{T_a-t} \left(we_t + B - \left(Q_{T_a|t} \beta^{T_a-t} R^{T_a-t} \right)^{-\frac{1}{\gamma}} \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} (\pi_l A_l + B) \right) \\
A_l &= \sum_{t=0}^{T_a} R^{T_a-t} \left(we_t + B - \left(Q_{T_a|t} \beta^{T_a-t} R^{T_a-t} \pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} (\pi_l A_l + B) \right) \\
A_l &= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + B) - (\pi_l A_l + B) \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} \sum_{t=0}^{T_a} \left(Q_{T_a|t} \beta^{T_a-t} R^{(1-\gamma)(T_a-t)} \right)^{-\frac{1}{\gamma}} \\
&\quad \left[1 + \pi_l \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} \sum_{t=0}^{T_a} \left(Q_{T_a|t} \beta^{T_a-t} R^{(1-\gamma)(T_a-t)} \right)^{-\frac{1}{\gamma}} \right] A_l \\
&= \sum_{t=0}^{T_a} R^{T_a-t} (we_t + B) - \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} \sum_{t=0}^{T_a} \left(Q_{T_a|t} \beta^{T_a-t} R^{(1-\gamma)(T_a-t)} \right)^{-\frac{1}{\gamma}} B
\end{aligned}$$

Let us define

$$\phi_l = \left(\pi_l \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} \quad (40)$$

$$\phi_b = \left(\pi_b \sum_{s=T_r}^T Q_{s|T_a} \beta^{s-T_a} \right)^{-\frac{1}{\gamma}} \quad (41)$$

and

$$\chi = \sum_{t=0}^{T_a} \left(Q_{T_a|t} \left(\beta R^{(1-\gamma)} \right)^{T_a-t} \right)^{-\frac{1}{\gamma}}. \quad (42)$$

$$F_e^{rr} = w \sum_{t=0}^{T_a} R^{T_a-t} e_t \quad (43)$$

is the time- T_a future value of preannuitization labor income while

$$f_B^{rr} = \sum_{t=0}^{T_a} R^{T_a-t} = \frac{R^{T_a} - 1}{r} \quad (44)$$

is the time- T_a future value multiplier of preannuitization bequest income.

$$A_l = \frac{F_e^{rr} + f_B^{rr} B - \phi_l \chi B}{1 + \pi_l \phi_l \chi}. \quad (45)$$

Likewise if we go to full bond annuitization, the bond annuity will be

$$A_b = \frac{F_e^{rr} + f_B^{rr} B - \phi_b \chi B}{1 + \pi_b \phi_b \chi}. \quad (46)$$

Let us focus on the full annuitization case. Define

$$A_0^l = \frac{F_e^{rr}}{1 + \pi_l \phi_l \chi} \quad (47)$$

$$A_1^l = \frac{f_B^{rr} - \phi_l \chi}{1 + \pi_l \phi_l \chi}, \quad (48)$$

so

$$A_l = A_0^l + A_1^l B.$$

For $t = 0, \dots, T_a$,

$$b_{t+1} = w e_t + R b_t + B - c_t$$

$$\begin{aligned} \sum_{s=0}^t R^{t-s} b_{s+1} &= \sum_{s=0}^t R^{t-s} (w e_s + R b_s + B - c_s) \\ \sum_{s=0}^t R^{t-s} b_{s+1} &= w \sum_{s=0}^t R^{t-s} e_s + \sum_{s=0}^t R^{t+1-s} b_s + \sum_{s=0}^t R^{t-s} B \\ &\quad - \sum_{s=0}^t R^{t-s} \left(Q_{T_a|s} \beta^{T_a-s} R^{T_a-s} \right)^{-\frac{1}{\gamma}} \phi_l (\pi_l A_0^l + (1 + \pi_l A_1^l) B) \end{aligned}$$

$$\begin{aligned} b_{t+1} + \sum_{s=0}^{t-1} R^{t-s} b_{s+1} &= w \sum_{s=0}^t R^{t-s} e_s + \sum_{s=1}^t R^{t+1-s} b_s + \sum_{s=0}^t R^{t-s} B \\ &\quad - \sum_{s=0}^t R^{t-T_a+T_a-s} \left(Q_{T_a|s} \beta^{T_a-s} R^{T_a-s} \right)^{-\frac{1}{\gamma}} \phi_l (\pi_l A_0^l + (1 + \pi_l A_1^l) B) \end{aligned}$$

$$\begin{aligned} b_{t+1} + \sum_{s=0}^{t-1} R^{t-s} b_{s+1} &= w \sum_{s=0}^t R^{t-s} e_s + \sum_{s=0}^{t-1} R^{t-s} b_{s+1} + \sum_{s=0}^t R^{t-s} B \\ &\quad - R^{t-T_a} \sum_{s=0}^t \left(Q_{T_a|s} (\beta R^{1-\gamma})^{T_a-s} \right)^{-\frac{1}{\gamma}} \phi_l (\pi_l A_0^l + (1 + \pi_l A_1^l) B) \end{aligned}$$

$$b_{t+1} = w \sum_{s=0}^t R^{t-s} e_s + \frac{R^{t+1} - 1}{r} B - \phi_l R^{t-T_a} \sum_{s=0}^t \left(Q_{T_a|s} (\beta R^{1-\gamma})^{T_a-s} \right)^{-\frac{1}{\gamma}} (\pi_l A_0^l + (1 + \pi_l A_1^l) B)$$

Let

$$\chi_t = \sum_{s=0}^t \left(Q_{T_a|s} (\beta R^{1-\gamma})^{T_a-s} \right)^{-\frac{1}{\gamma}} \quad (49)$$

and

$$H_t = w \sum_{s=0}^t R^{t-s} e_s$$

Then

$$b_{t+1} = H_t + \frac{R^{t+1} - 1}{r} B - \phi_l R^{t-T_a} \chi_t (\pi_l A_0^l + (1 + \pi_l A_1^l) B)$$

for $t = 0, \dots, T_a - 1$. The bequest balance equation is

$$\begin{aligned} PB &= R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[H_t + \frac{R^{t+1} - 1}{r} B - \phi_\pi R^{t-T_a} \chi_t (\pi A_0 + (1 + \pi A_1) B) \right] \\ &= \left(P - R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[\frac{R^{t+1} - 1}{r} - \phi_l R^{t-T_a} \chi_t (1 + \pi_l A_1^l) \right] \right) B \\ &= R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) [H_t - \phi_l R^{t-T_a} \chi_t \pi_l A_0^l]. \end{aligned}$$

In general equilibrium with full annuitization, the bequest will be

$$B = \frac{R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) [H_t - \phi_l R^{t-T_a} \chi_t \pi_l A_0^l]}{P - R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[\frac{R^{t+1} - 1}{r} - \phi_l R^{t-T_a} \chi_t (1 + \pi_l A_1^l) \right]} \quad (50)$$

Likewise, if we enforce full bond annuitization, we can define

$$A_0^b = \frac{F_e^{rr}}{1 + \pi_b \phi_b \chi} \quad (51)$$

$$A_1^b = \frac{f_B^{rr} - \phi_b \chi}{1 + \pi_b \phi_b \chi}, \quad (52)$$

so

$$A^b = A_0^b + A_1^b B.$$

For $t = 0, \dots, T_a$,

$$\begin{aligned} b_{t+1} &= \sum_{s=0}^t R^{t-s} [w e_s + B - c_s] \\ b_{t+1} &= H_t^a + \frac{R^{t+1} - 1}{r} B - \phi_b R^{t-T_a} \chi_t (\pi_b A_0^b + (1 + \pi_b A_1^b) B). \end{aligned} \quad (53)$$

For $t = T_r, \dots, T + 1$,

$$b_t = R^{t-T_r} \left(1 - \frac{\pi_b}{r} \right) A^b + \frac{\pi_b}{r} A^b$$

$$b_{t+1} = \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] (A_0^b + A_1^b B).$$

The bequest balance equation becomes

$$\begin{aligned} PB &= R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[H_t + \frac{R^{t+1} - 1}{r} B - \phi_b R^{t-T_a} \chi_t (\pi_b A_0^b + (1 + \pi_b A_1^b) B) \right] \\ &\quad + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] (A_0^b + A_1^b B) \\ &= \left[\begin{array}{l} P - R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[\frac{R^{t+1} - 1}{r} - \phi_b R^{t-T_a} \chi_t (1 + \pi_b A_1^b) \right] \\ - R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] A_1^b \end{array} \right] B \\ &= R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) [H_t^a - \phi_b R^{t-T_a} \chi_t \pi_b A_0^b] + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] A_0^b \end{aligned}$$

In general equilibrium with full bond annuitization,

$$B = \frac{R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) [H_t - \phi_b R^{t-T_a} \chi_t \pi_b A_0^b] + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] A_0^b}{P - R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) \left[\frac{R^{t+1} - 1}{r} - \phi_b R^{t-T_a} \chi_t (1 + \pi_b A_1^b) \right] - R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \left[\frac{\pi_b}{r} + R^{t-T_a} \left(1 - \frac{\pi_b}{r} \right) \right] A_1^b} \quad (54)$$

2.1 Numerical Results

With rational households, the portfolio allocation regarding investment vehicles with a certain return will be determined entirely by the rate of return. Households will invest exclusively in whatever vehicle available to them offers the highest return. Perfect Yaari (1969) annuities that pay a return equal to the net return on capital r plus the hazard rate of dying would be the best possible annuity in terms of return, but these are rare in reality. What we have termed longevity annuities above would offer the next highest return π_l . Bond annuities and regular bonds would offer the lowest return of r . To better understand the results from equilibria where households are forced to invest exclusively in annuities, what we term the Full Annuitization case, or only in bond annuities, we also provide results from equilibria where households have access to Yaari annuities (the Perfect Annuitization case) and where households have no access to annuities (the No Annuitization case).⁷

Since most households do not annuitize, we calibrate the model to match macroeconomic observables in the No Annuitization case with a share of capital $\alpha = 0.3375$, a capital-output ratio $K/Y = 3.0$, and a consumption-output ratio $C/Y = 0.7$. These targets for K/Y and C/Y imply a depreciation rate $\delta = 0.10$. For each case, we also consider five different values of the intertemporal elasticity with $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 1.0$, $\gamma = 2.0$, and $\gamma = 3.0$. For each choice of γ , the discount factor β is chosen to achieve the target value of $K/Y = 3.0$ in the No Annuitization case. These are reported in Table 1. Note that higher values of γ require lower values of β to achieve the same K/Y . The productivity profile e_t is taken from Gourinchas and Parker (2002) and the mortality portfolio Q_t is taken from Feigenbaum (2008).

Fig. 1 shows how K/Y varies with γ for each of the four investment cases. For very low γ , K/Y is highest in the Perfect Annuitization case and lowest in the Bond Annuity case. However, as γ increases (and β decreases), K/Y increases in the Bond Annuity case while it decreases in the Perfect and Full Annuitization

⁷See Feigenbaum, Gahramanov, and Tang (2013) for details of the Perfect and No Annuitization regimes.

γ	0.25	0.5	1.0	2.0	3.0
β	0.978	0.977	0.973	0.962	0.951

Table 1: Discount factors corresponding to each intertemporal elasticity.

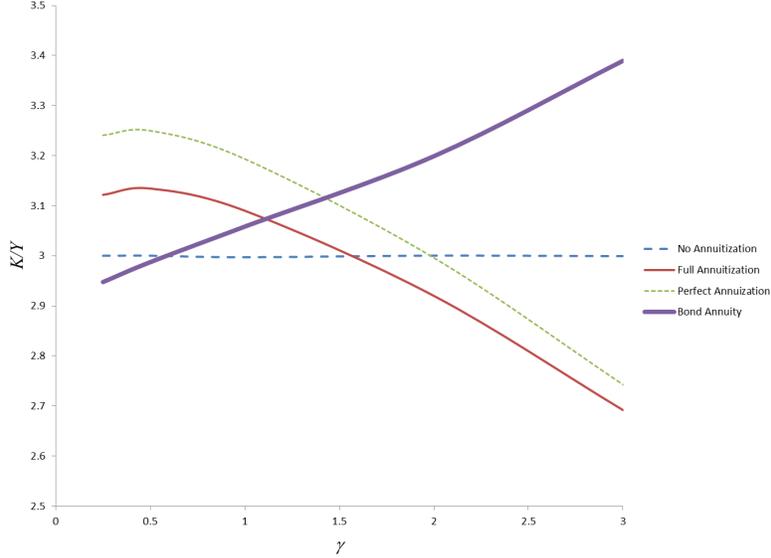


Figure 1: Equilibrium K/Y as a function of the inverse intertemporal elasticity γ under four assumptions about portfolio composition.

cases. Thus for $\gamma > 2$, the Bond Annuity case has the highest K/Y while the Full Annuitization case has the lowest with the Perfect Annuitization K/Y midway between the No Annuitization target of 3.0 and the lower Full Annuitization value.

Fig. 2 shows how the compensating variation Δ varies with γ . For case i , this is the fraction by which consumption in the No Annuitization case needs to be augmented in order to achieve the same lifetime utility as in case i :

$$\sum_{t=0}^T Q_t \beta^t u((1 + \Delta)c_t^{NA}) = \sum_{t=0}^T Q_t \beta^t u(c_t^i).$$

Qualitatively Fig. 2 looks very much like Fig. 1, although the Bond Annuity case has the highest utility for γ as low as 0.5. To help understand the behavior of the compensating variation with respect to γ , we plot the lifecycle consumption profiles for the baseline $\gamma = 1$ (Fig. 3) and the extreme values of $\gamma = 3$ (Fig. 4) and $\gamma = 0.25$ (Fig. 5). For $\gamma = 1$, the Bond Annuity case has the highest consumption profile prior to retirement. After retirement, the Bond Annuity profile flattens out at a value much less than the Perfect and Full Annuitization profiles, but utility at retirement is discounted by a factor of 0.33. The Full Annuitization and Perfect Annuitization profiles are nearly identical except the Full Annuitization profile is smoother after retirement. The No Annuitization case is in between the Bond Annuity profile and the two regular Annuitization profiles prior to retirement, which explains why it provides utility intermediate between these two groupings. It underperforms the Bond Annuity case primarily because it has the least smooth profile after retirement.

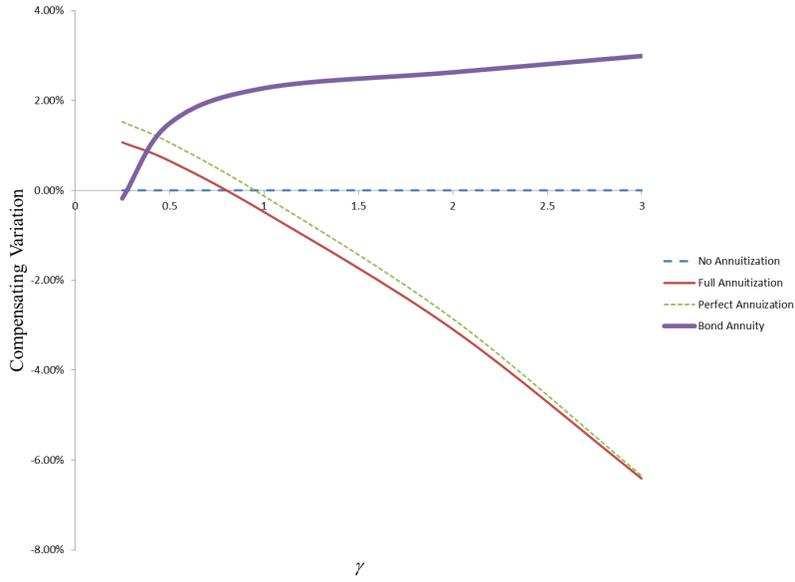


Figure 2: General-equilibrium compensating variation relative to unrestricted bond case as a function of the inverse intertemporal elasticity γ under four assumptions about portfolio composition.

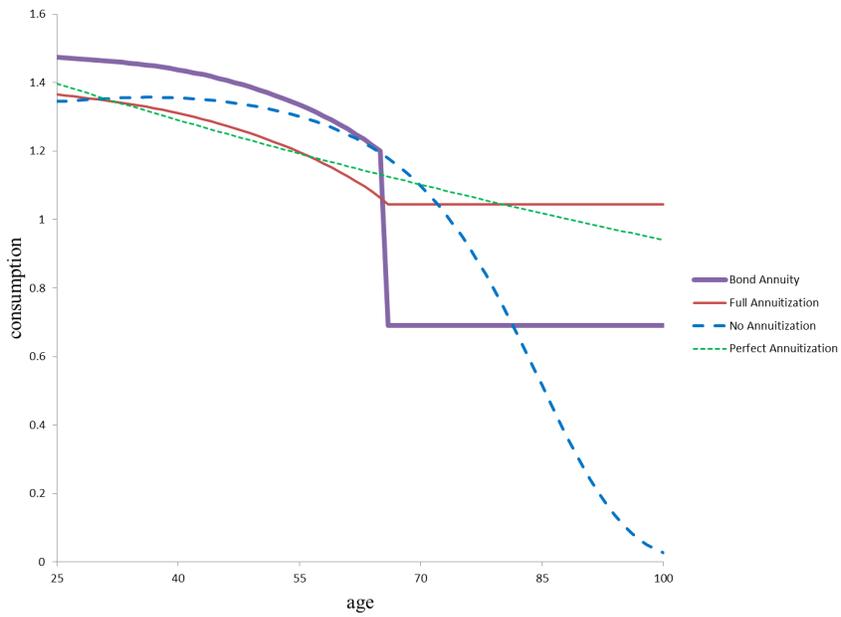


Figure 3: Lifecycle consumption profiles for $\gamma = 1$ calibration under four assumptions about portfolio composition.

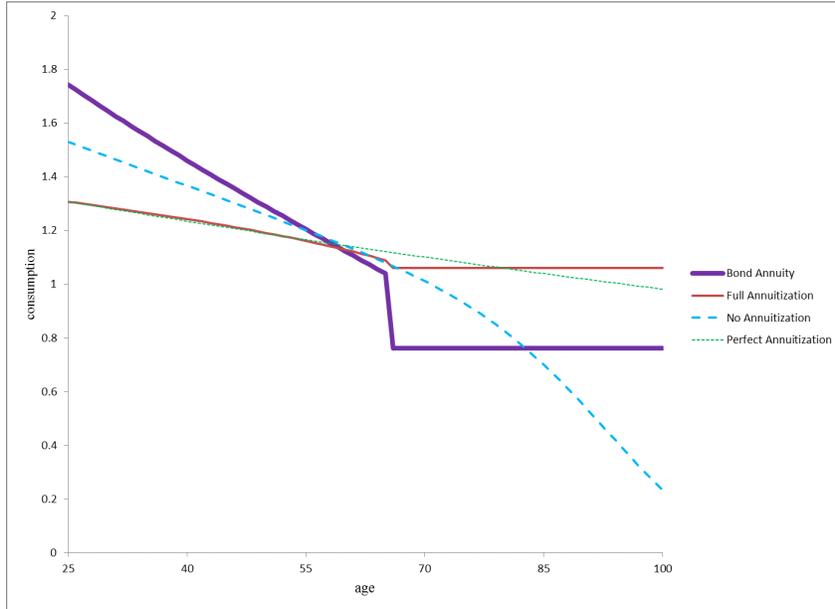


Figure 4: Lifecycle consumption profiles for $\gamma = 3$ calibration under four assumptions about portfolio composition.

Qualitatively, the profiles for $\gamma = 3$, in Fig. 4, are quite similar to the profiles for $\gamma = 1$, and the utility ranking of the four cases is also the same. On the other hand, we get strikingly different behavior when the elasticity of intertemporal substitution is very high, as in Fig. 5 with $\gamma = 0.25$. In the absence of perfect annuitization the consumption profiles depend on the mortality profile. The elasticity of consumption growth with respect to the hazard rate of dying is the elasticity of intertemporal substitution, so for high γ the effect of this dependence is suppressed. But for low γ , the dependence is blatantly apparent. The profiles without perfect annuitization are not smooth. The Bond Annuity profile is the least smooth, starting at the lowest initial value, attaining the highest maximum value just before retirement, and then going to a very low value again after retirement. Consequently, the Bond Annuity profile delivers the least utility when $\gamma = 0.25$. In contrast, the smooth Perfect Annuitization profile gives the highest utility.

Most macroeconomic research assumes a value of γ between 1 and 3, for which the Bond Annuity case is best of the four we consider here. For $\gamma = 1$ the Bond Annuity case offers the equivalent of 2% more consumption over the lifecycle than the other three cases. Vidangos (2008) estimates that the benefit of eliminating idiosyncratic income risk would be equivalent to increasing consumption by only 1%, so the benefit of getting households to invest in bond annuities would be quite significant. For larger values of γ , the benefits of bond annuities continue to increase marginally. The Perfect and Full Annuitization cases deliver roughly the same utility for $\gamma \geq 2$, which decreases sharply. For $\gamma = 2$, they give the equivalent of 3% less consumption than the No Annuitization case.

Our intuition is correct that utility will be maximized with perfect annuitization and that tying a household's hands by forcing them to take a constant stream of consumption is bad for utility, but only in a partial equilibrium where factor prices and the bequest B are the same in all four cases. To see this we plot the compensating variations in partial equilibrium in Fig. 6 with factor prices and the bequest set at their No Annuitization values. One might think that what is driving utility to be higher in the no annuitization

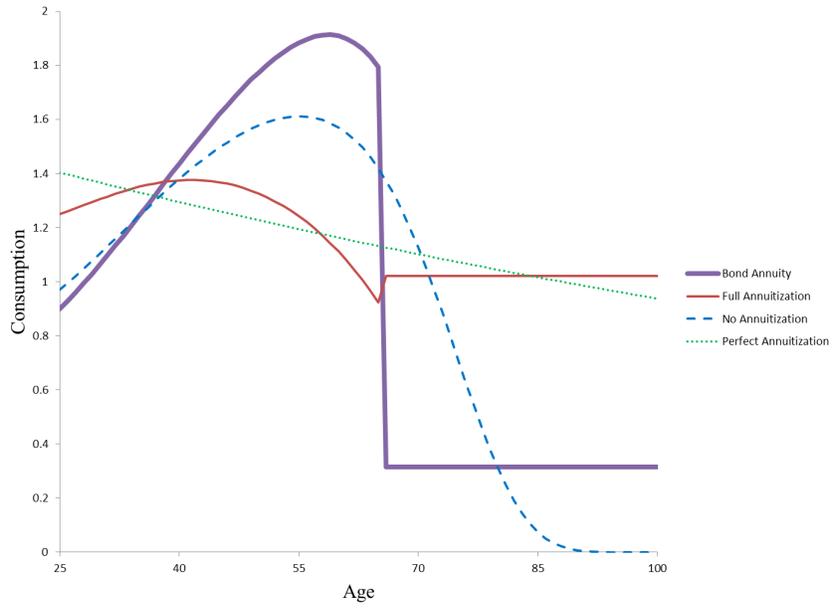


Figure 5: Lifecycle consumption profiles for $\gamma = 0.25$ calibration under four assumptions about portfolio composition.

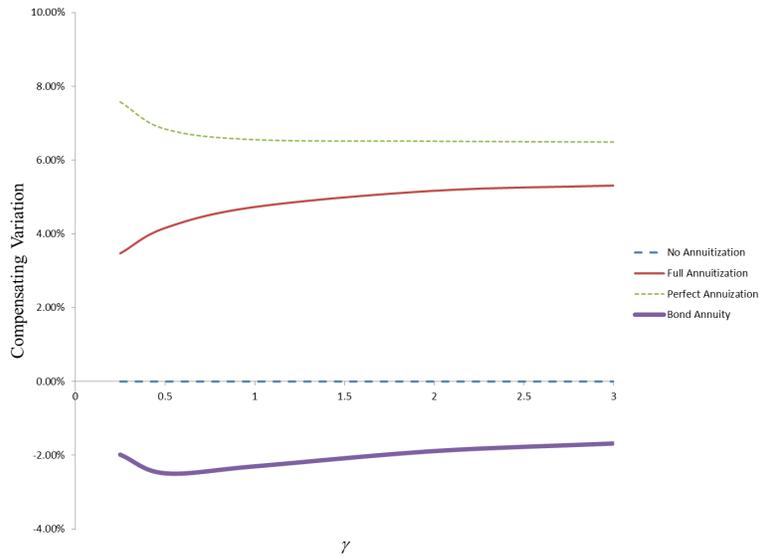


Figure 6: Partial-equilibrium compensating variation relative to unrestricted bond case as a function of the inverse intertemporal elasticity γ under four assumptions about portfolio composition.

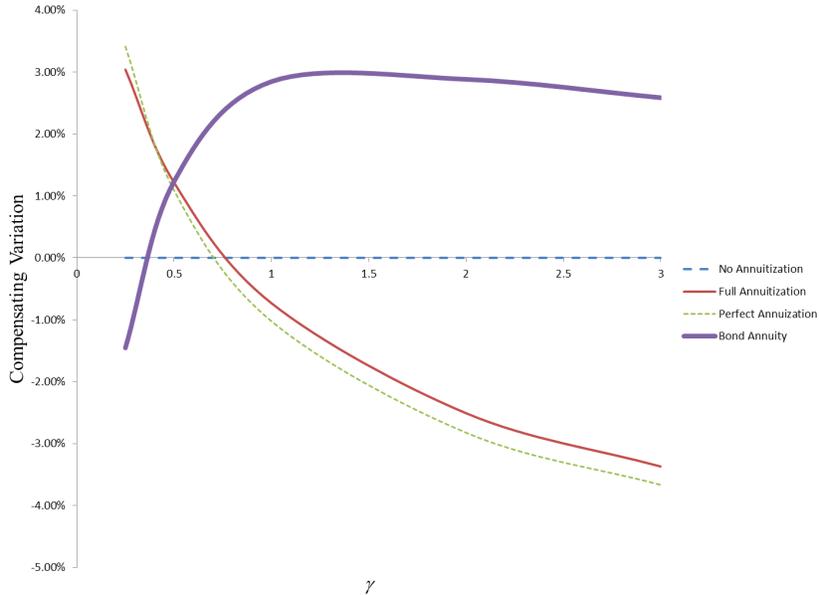


Figure 7: Quasipartial-equilibrium compensating variation relative to unrestricted bond case as a function of the inverse intertemporal elasticity γ under four assumptions about portfolio composition.

and even higher in the Bond Annuity cases is that bequests allow households to save more and generate a higher capital stock. The similarity between Figs. 1 and 2 also lends itself to the explanation that the Bond Annuity yields more advantageous factor prices. In fact, however, the same pattern of compensating variations holds even if we control for factor prices. This can be seen in Fig. 7, which shows the compensating variations for quasipartial equilibria where we enforce the bequest balance equation (21) while holding factor prices at their No Annuity values. Utility is highest in the Bond Annuity case and lowest in the Perfect Annuity case primarily because bequests allow households to achieve better consumption streams even when factor prices are fixed. For most calibrations, the equilibrium with bond annuitization is better than the equilibrium with full annuitization. Usually the equilibrium with no restrictions and no annuities lies in between. Annuity provides a smoother consumption stream. When there is a bequest, it is better to have the smooth retirement consumption than to face the possibility of low consumption late in life. Bond annuities provide longevity insurance while also allowing for bequests. The explanation for this needs to be fleshed out.

3 Restricted Optimal Irrational Behavior

Next let us consider what happens if households follow a Keynesian consumption rule

$$c_t = mwe_t + c_{aut} \tag{55}$$

as in Feigenbaum, Gahramanov, and Tang (FGT) (2013a), where $m \in [0, 1]$ is the marginal propensity to consume and $c_{aut} > 0$ is autonomous consumption. We also assume that households invest a fixed

amount A of their saving at retirement as an annuity. Following the restricted optimal irrational behavior (ROIB) paradigm of Feigenbaum, Gahramanov, and Tang (2013a,b), we assume that m , c_{aut} , A , and K are determined so as to maximize U in general equilibrium.

Thus we wish to solve

$$\max_{m, c_{aut}, A} \sum_{t=0}^T \beta^t Q_t u(c_t)$$

subject to (55), (2)-(7), (21)-(20), and (24)-(26).

Note that if $c_t = c_{aut}$ for $t > T_r$,

$$c_{aut} = Rb_t + \pi_l A_l + \pi_b A_b + B - b_{t+1}.$$

This can be achieved either by any investment strategy that satisfies

$$Rb_t - b_{t+1} + \pi_b A_b = c_{aut} - \pi_l A_l - B,$$

which can be achieved either by investing in the bond annuity or by replicating the bond annuity with the household's individual bond investments. Since the bond annuity is financed through ordinary bond investments, the two approaches will have completely equivalent effects. For simplicity, we assume that the household sets $b_{t+1} = 0$ for $t > T_a$ and invests in the bond annuity so that

$$A_b = \frac{c_{aut} - \pi_l A_l - B}{\pi_b}.$$

The question then is what is the optimal portfolio allocation between the longevity annuity and the bond annuity.

If we insert (55) into (36), we get

$$\begin{aligned} \sum_{t=0}^T \frac{mwe_t + c_{aut}}{R^t} &= \sum_{t=0}^T \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} \\ \left[1 - \frac{1}{R^{T+1}}\right] \frac{R}{r} c_{aut} &= (1-m) \sum_{t=0}^T \frac{we_t}{R^t} + \left(1 - \frac{1}{R^{T+1}}\right) \frac{RB}{r} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} \end{aligned}$$

Autonomous consumption is

$$c_{aut} = \frac{1}{1 - R^{-T-1}} \frac{r}{R} \left((1-m) \sum_{t=0}^T \frac{we_t}{R^t} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} \right) + B. \quad (56)$$

For $t \in [0, T_a)$, (27) simplifies to

$$b_{t+1} = R^t \sum_{s=0}^t \frac{(1-m)we_s + B - c_{aut}}{R^s}. \quad (57)$$

The bequest balance equation (21) becomes

$$BP = R \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1})b_{t+1} + R \sum_{t=T_a}^T (Q_t - Q_{t+1})a_{t+1}^b$$

Using (15), this becomes

$$\begin{aligned}
BP &= \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) R^{t+1} \sum_{s=0}^t \frac{(1-m)we_s + B - c_{aut}}{R^s} \\
&\quad + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \frac{1 - R^{-(T-t)}}{1 - R^{-(T-T_a)}} A_b \\
BP &= \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) R^{t-s+1} \sum_{s=0}^t [(1-m)we_s + B - c_{aut}] \\
&\quad + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) \frac{1 - R^{t-T}}{r} (c_{aut} - \pi_l A_l - B)
\end{aligned} \tag{58}$$

Note that we can rewrite (56) and (58) to obtain

$$\begin{aligned}
B &= \frac{1}{P} \left\{ (1-m)w \sum_{t=0}^{T_a-1} \sum_{s=0}^t (Q_t - Q_{t+1}) R^{t-s+1} e_s - \pi_l A_l \frac{R}{r} \sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) \right. \\
&\quad \left. + (c_{aut} - B) \left[\frac{R}{r} \sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) - \sum_{t=0}^{T_a-1} \sum_{s=0}^t (Q_t - Q_{t+1}) R^{t-s+1} \right] \right\} \\
B &= \frac{1}{P} \left\{ (1-m)w \sum_{t=0}^{T_a-1} \sum_{s=0}^t (Q_t - Q_{t+1}) R^{t-s+1} e_s - \pi_l A_l \frac{R}{r} \sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) \right. \\
&\quad \left. + (c_{aut} - B) \left[\frac{R}{r} \sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) - \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) R^{t+1} \frac{R^{-(t+1)} - 1}{R^{-1} - 1} \right] \right\} \\
B &= \frac{1}{P} \left\{ (1-m)w \sum_{t=0}^{T_a-1} \sum_{s=0}^t (Q_t - Q_{t+1}) R^{t-s+1} e_s - \pi_l A_l \frac{R}{r} \sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) \right. \\
&\quad \left. + (c_{aut} - B) \left[\frac{R}{r} \sum_{t=T_r}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) - \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1}) R \frac{1 - R^{t+1}}{1 - R} \right] \right\}
\end{aligned}$$

Thus we obtain the equations

$$c_{aut} - B = \frac{1}{1 - R^{-T-1}} \frac{r}{R} \left((1-m) \sum_{t=0}^T \frac{we_t}{R^t} + \left[\frac{1 - \frac{1}{R^{T-T_a}}}{r} \pi_l - 1 \right] \frac{A_l}{R^{T_a}} \right) \tag{59}$$

$$\begin{aligned}
B &= \frac{1}{P} \left\{ (1-m)w \sum_{t=0}^{T_a} \sum_{s=0}^t (Q_t - Q_{t+1}) R^{t-s+1} e_s - \pi_l A_l \frac{R}{r} \sum_{t=T_r}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) \right. \\
&\quad \left. + (c_{aut} - B) \frac{R}{r} \left[\sum_{t=T_a}^T (Q_t - Q_{t+1}) (1 - R^{t-T}) - \sum_{t=0}^{T_a} (Q_t - Q_{t+1}) (R^{t+1} - 1) \right] \right\}
\end{aligned} \tag{60}$$

Given m , A , and K , (59) and (60) give us c_{aut} and B .

Let us rewrite these equations in terms of variables that depend only on exogenous variables, A , or K .

$$H(K) = w(K) \sum_{t=0}^T \frac{e_t}{R(K)^t} \quad (61)$$

is the present value of the household's income while

$$H(t, K) = w(K) \sum_{s=0}^t \frac{e_s}{R(K)^s} \quad (62)$$

is the present value of income prior to age t . The longevity-annuity payoff ratio as a function of K is

$$\pi_l(K) = \frac{1}{\sum_{s=T_a+1}^T \frac{Q_s^{T_a}}{R(K)^{s-T_a}}} \quad (63)$$

$$\Delta = c_{aut} - B = \frac{1}{1 - R(K)^{-T-1}} \frac{r(K)}{R(K)} \left((1 - m)H(K) + \left[\frac{1 - \frac{1}{R(K)^{T-T_a}}}{r(K)} \pi_l(K) - 1 \right] \frac{A}{R(K)^{T_a}} \right) \quad (64)$$

$$B_r(K) = \frac{R(K)}{r(K)} \sum_{t=T_a}^T (Q_t - Q_{t+1})(1 - R(K)^{t-T}) \quad (65)$$

$$B_w(K) = \frac{R(K)}{r(K)} \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1})(R(K)^{t+1} - 1) \quad (66)$$

Finally, we define

$$H_B(K) = \sum_{t=0}^{T_a-1} (Q_t - Q_{t+1})R^{t+1}H(t, K). \quad (67)$$

Thus we can rewrite

$$B = \frac{1}{P} \{ (1 - m)H_B(K) - \pi_l(K)A_l B_r(K) + [B_r(K) - B_w(K)]\Delta \}$$

Note that since $H_B(K)$, $B_w(K)$, $B_r(K)$, and $H(K)$ do not depend on m , c_{aut} and B are both linear functions of m :

$$\Delta = \Delta_0 + \Delta_1 m$$

and

$$B = B_0 + B_1 m.$$

$$\Delta_0 = \frac{1}{1 - R(K)^{-T-1}} \frac{r(K)}{R(K)} \left(H(K) + \left[\frac{1 - \frac{1}{R(K)^{T-T_a}}}{r(K)} \pi_l(K) - 1 \right] \frac{A_l}{R(K)^{T_a}} \right) \quad (68)$$

$$\Delta_1 = -\frac{1}{1 - R(K)^{-T-1}} \frac{r(K)}{R(K)} H(K) \quad (69)$$

$$B_0 = \frac{1}{P} \{ H_B(K) - \pi_l(K)A_l B_r(K) + [B_r(K) - B_w(K)]\Delta_0 \} \quad (70)$$

$$B_1 = \frac{-H_B(K) + [B_r(K) - B_w(K)]\Delta_1}{P} \quad (71)$$

The capital condition determines m .

$$K = \sum_{t=0}^T Q_t b_{t+1} + K_a$$

$$\begin{aligned} K - K_a &= \sum_{t=0}^{T_a-1} Q_t b_{t+1} + \sum_{t=T_a}^T Q_t a_{t+1}^b \\ &= \sum_{t=0}^{T_a-1} \sum_{s=0}^t Q_t R^{t-s} [(1-m)w e_s - \Delta] + \sum_{t=T_a}^T Q_t \frac{1-R^{t-T}}{r} (\Delta - \pi_l A_l) \\ &= (1-m) \sum_{t=0}^{T_a-1} Q_t R^t H(t, K) - \Delta \sum_{t=0}^{T_a-1} Q_t R^t \frac{1-R^{-t-1}}{1-R^{-1}} + \sum_{t=T_a}^T Q_t \frac{1-R^{t-T}}{r} (\Delta - \pi_l A_l) \\ &= (1-m) \sum_{t=0}^{T_a-1} Q_t R^t H(t, K) - \Delta \sum_{t=0}^{T_a-1} Q_t R^{t+1} \frac{1-R^{-t-1}}{r} + \sum_{t=T_a}^T Q_t \frac{1-R^{t-T}}{r} (\Delta - \pi_l A_l) \\ &= (1-m) \sum_{t=0}^{T_a-1} Q_t R^t H(t, K) - \Delta \sum_{t=0}^{T_a-1} Q_t \frac{R^{t+1} - 1}{r} + (\Delta - \pi_l A_l) \sum_{t=T_a}^T Q_t \frac{1-R^{t-T}}{r} \end{aligned}$$

$$S_r(K) = \frac{1}{r(K)} \sum_{t=T_a}^T Q_t (1-R(K)^{t-T}) \quad (72)$$

$$S_w(K) = \frac{1}{r(K)} \sum_{t=0}^{T_a-1} Q_t (R(K)^{t+1} - 1) \quad (73)$$

Finally, we define

$$H_S(K) = \sum_{t=0}^{T_a-1} Q_t R^t H(t, K). \quad (74)$$

Thus we can write the capital equation as

$$\begin{aligned} K - K_a &= (1-m)H_S(K) - \Delta S_w(K) + (\Delta - \pi_l A_l) S_r(K) \\ K - K_a &= (1-m)H_S(K) + (S_r(K) - S_w(K))(\Delta_0 + \Delta_1 m) - \pi_l A_l S_r(K) \\ K - K_a - H_S(K) + \pi_l A_l S_r(K) - (S_r(K) - S_w(K))\Delta_0 &= [(S_r(K) - S_w(K))\Delta_1 - H_S(K)]m \\ m(K) &= \frac{K - K_a - H_S(K) + \pi_l A_l S_r(K) - (S_r(K) - S_w(K))\Delta_0}{(S_r(K) - S_w(K))\Delta_1 - H_S(K)} \end{aligned} \quad (75)$$

Thus m , c_{aut} , and B can all be expressed as functions of K and A_l . The ROIB behavior is determined by computing what K and A_l maximizes U .

The ROIB paradigm differs significantly from the rational paradigm in that the action of preferences is completely separated from what behaviors are feasible. The set of feasible market equilibria with Keynesian consumption functions is completely defined by $m(K, A_l)$, $c_{aut}(K, A_l)$, and $B(K, A_l)$. Preferences only matter for determining which of these equilibria is optimal.

Depending on how fine a grid of (K, A_l) points one considers, there may be very many equilibria. For each one, it is necessary to store the sequence $\{c_t\}_{t=0}^T$ to compute the utility for each preference model. Thus, estimating what preferences are most consistent with an observed set of behavior requires a big data approach under the ROIB paradigm.

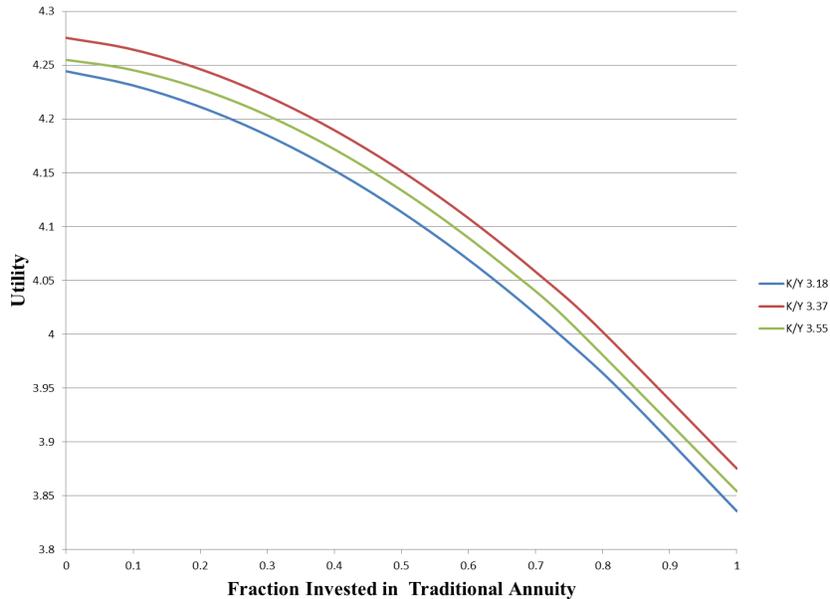


Figure 8: Lifetime utility as a function of traditional annuitization fraction under the ROIB paradigm for three technology calibrations and the baseline preference calibration.

3.1 Numerical Results

We maintain the same technological parameters as in Section 2.1. For our baseline calibration, we consider $\beta = 0.96$ and $\gamma = 1$. Unlike in the rational paradigm, we can freely choose the capital stock K and the fraction of retirement saving put in the longevity annuity

$$\phi_l = \frac{A_l}{A_b + A_l},$$

using (75) to determine the marginal propensity to consume m . Fig. 8 shows how lifetime utility varies with the annuitization fraction for the optimal K/Y of 3.37 and for two values to either side of this optimum. For the baseline calibration, utility is maximized when the household puts all of its retirement savings into the bond annuity, leaving nothing for the longevity annuity. This also occurs for lower discount factors.

Only for very high discount factors will households be better off participating in the longevity annuity. Fig. 9 is the same graph with a slightly higher discount factor of $\beta = 0.973$. Here we see that utility is maximized if it invests 60% of retirement savings in the longevity annuity.

4 Fully Rational Households

The calculations are actually most difficult for the model regime where we have a rational household with no restrictions on its consumption. The coding of this model remains incomplete, and we have no results to report as of yet.

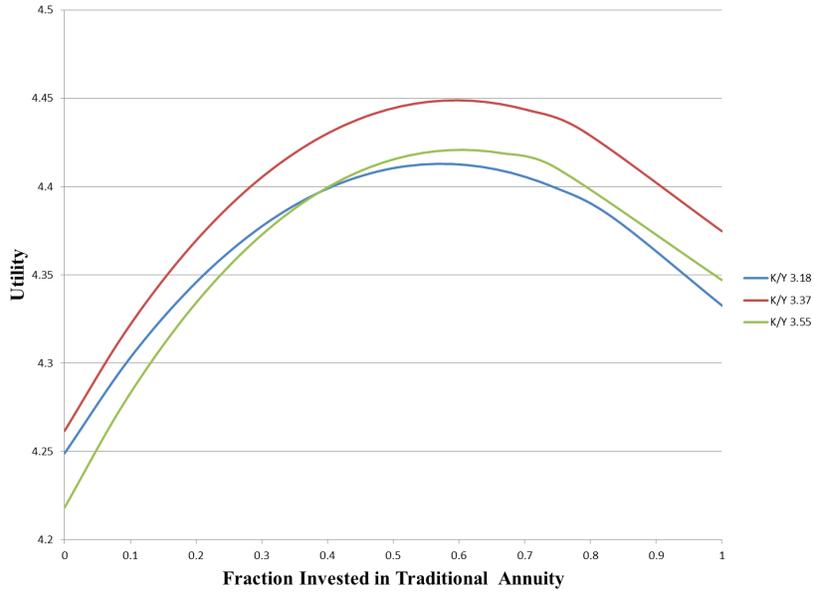


Figure 9: Lifetime utility as a function of traditional annuitization fraction under the ROIB paradigm for three technology calibrations and $\beta = 0.973$.

A fully rational household will maximize

$$U = \sum_{t=0}^T Q_t \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + b_{t+1} &= we_t + Rb_t + B \quad t \in [0, T_a) \\ c_{T_a} + b_{T_r} + A_l + A_b &= we_{T_a} + Rb_{T_a} + B \\ c_t + b_{t+1} &= Rb_t + \pi_l A_l + \pi_b A_b + B \quad t \in [T_r, T] \\ b_{t+1} &\geq 0 \quad t \in [T_a, T) \\ A_l, A_b &\geq 0 \end{aligned}$$

and

$$b_0 = b_{T+1} = 0.$$

This problem has the Lagrangian

$$\begin{aligned}
\mathcal{L} = & \sum_{t=0}^T Q_t \beta^t u(c_t) + \sum_{t=0}^{T_a-1} \lambda_t [we_t + Rb_t + B - c_t - b_{t+1}] \\
& + \lambda_{T_a} [we_{T_a} + Rb_{T_a} + B - c_{T_a} - b_{T_r} - A_l - A_b] \\
& + \sum_{t=T_r}^T \lambda_t [Rb_t + \pi_l A_l + \pi_b A_b + B - c_t - b_{t+1}] \\
& + \sum_{t=T_r}^T \mu_t b_t + \nu_l A_l + \nu_b A_b
\end{aligned} \tag{76}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = Q_t \beta^t u'(c_t) - \lambda_t = 0 \tag{77}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = -\lambda_t + R\lambda_{t+1} = 0 \quad t = 1, \dots, T_a - 1 \tag{78}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = -\lambda_t + R\lambda_{t+1} + \mu_{t+1} = 0 \quad t = T_a, \dots, T - 1 \tag{79}$$

$$\frac{\partial \mathcal{L}}{\partial A_l} = -\lambda_{T_a} + \pi_l \sum_{t=T_r}^T \lambda_t + \nu_l = 0 \tag{80}$$

$$\frac{\partial \mathcal{L}}{\partial A_b} = -\lambda_{T_a} + \pi_b \sum_{t=T_r}^T \lambda_t + \nu_b = 0 \tag{81}$$

We have

$$\lambda_t = Q_t \beta^t u'(c_t). \tag{82}$$

Combining (80) and (81), we get

$$\begin{aligned}
\pi_l \sum_{t=T_r}^T \lambda_t + \nu_l &= \pi_b \sum_{t=T_r}^T \lambda_t + \nu_b. \\
\nu_b - \nu_l &= (\pi_l - \pi_b) \sum_{t=T_r}^T \lambda_t
\end{aligned}$$

Since $\pi_l > \pi_b$ and $\lambda_t > 0$, the right-hand side is positive. Therefore $\nu_b > \nu_l \geq 0$, so we must have $A_b = 0$. Inserting (82) into (80), we obtain

$$Q_{T_a} \beta^{T_a} u'(c_{T_a}) \geq \pi_l \sum_{t=T_r}^T Q_t \beta^t u'(c_t),$$

and the inequality is only possible if $A_l = 0$. This simplifies to

$$1 \geq \pi_l \sum_{t=T_r}^T Q_{t|T_a} \beta^{t-T_a} \frac{u'(c_t)}{u'(c_{T_a})} \tag{83}$$

with equality if $A_l > 0$. If none of the bond borrowing constraints bind,

$$\lambda_t = R^{-t}\lambda_0. \quad (84)$$

$$1 \geq \pi_l \sum_{t=T_r}^T \frac{\lambda_t}{\lambda_{T_a}} = \pi_l \sum_{t=T_r}^T \frac{R^{-t}\lambda_0}{R^{-T_a}\lambda_0} = \pi_l \sum_{t=T_r}^T R^{-(t-T_a)} = \frac{\pi_l}{\pi_b} > 1.$$

This is a contradiction, so at least one of the borrowing constraints must bind. For $A_l \geq 0$, let

$$V(A_l) = \max \sum_{t=0}^T Q_t \beta^t u(c_t) \quad (85)$$

subject to

$$\begin{aligned} c_t + b_{t+1} &= we_t + Rb_t + B \quad t \in [0, T_a) \\ c_{T_a} + b_{T_r} + A_l &= we_{T_a} + Rb_{T_a} + B \\ c_t + b_{t+1} &= Rb_t + \pi_l A_l + B \quad t \in [T_r, T] \\ b_{t+1} &\geq 0 \quad t \in [T_a, T) \\ A_l &\geq 0 \end{aligned}$$

and

$$b_0 = b_{T+1} = 0.$$

By the Envelope Theorem, for $A > 0$,

$$\frac{dV}{dA_l} = \frac{\partial \mathcal{L}}{\partial A_l} = -Q_{T_a} \beta^{T_a} u'(c_{T_a}) + \pi \sum_{t=T_r}^T Q_t \beta^t u'(c_t). \quad (86)$$

The Euler equation is (78) for $t = 0, \dots, T_a - 1$:

$$Q_t \beta^t u'(c_t) = Q_{t+1} \beta^{t+1} R u'(c_{t+1}),$$

which yields

$$u'(c_t) = \frac{Q_{t+1}}{Q_t} \beta R u'(c_{t+1}). \quad (87)$$

For $t = T_a, \dots, T - 1$, the Euler equation is

$$Q_t \beta^t u'(c_t) = Q_{t+1} \beta^{t+1} R u'(c_{t+1}) + \mu_{t+1}.$$

Thus we have

$$u'(c_t) \geq \frac{Q_{t+1}}{Q_t} \beta R u'(c_{t+1}) \quad (88)$$

with equality if $b_{t+1} > 0$.

We are maximizing over consumption and saving paths and annuities that satisfy (2)-(7). The set of feasible consumption and saving paths will be Σ , which includes $\left(\{c_t\}_{t=0}^T, \{b_{t+1}\}_{t=0}^T, A \right)$ such that $A_l \geq 0$,

$$0 \leq c_t \leq we_t + Rb_{T_a} + B - b_{t+1} \quad t \in [0, T_a)$$

$$0 \leq c_{T_a} \leq we_{T_a} + Rb_{T_a} + B - b_{T_r} - A_l$$

$$\begin{aligned} 0 \leq c_t &\leq \pi_l A_l + B + Rb_t - b_{t+1} \quad t \in [T_r, T] \\ b_t &\geq 0 \quad t \in [T_r, T], \end{aligned}$$

and

$$b_0 = b_{T+1} = 0.$$

Let $\left(\{c_t^1\}_{t=0}^T, \{b_{t+1}^1\}_{t=0}^T, A_l^1\right)$ and $\left(\{c_t^2\}_{t=0}^T, \{b_{t+1}^2\}_{t=0}^T, A_l^2\right)$ be feasible paths and let $\sigma \in [0, 1]$. Let

$$c_t^\sigma = \sigma c_t^1 + (1 - \sigma)c_t^2, \quad (89)$$

$$b_t^\sigma = \sigma b_t^1 + (1 - \sigma)b_t^2, \quad (90)$$

and

$$A_l^\sigma = \sigma A_l^1 + (1 - \sigma)A_l^2. \quad (91)$$

Then

$$\begin{aligned} 0 \leq c_t^1 &\leq we_t + Rb_t^1 + B - b_{t+1}^1 \quad t \in [0, T_a) \\ 0 \leq c_t^2 &\leq we_t + Rb_t^2 + B - b_{t+1}^2 \quad t \in [0, T_a) \\ 0 \leq c_{T_a}^1 &\leq we_{T_a} + Rb_{T_a}^1 + B - b_{T_r}^1 - A_l^1 \\ 0 \leq c_{T_a}^2 &\leq we_{T_a} + Rb_{T_a}^2 + B - b_{T_r}^2 - A_l^2 \\ 0 \leq c_t^1 &\leq \pi_l A_l^1 + B + Rb_t^1 - b_{t+1}^1 \quad t \in [T_r, T] \\ 0 \leq c_t^2 &\leq \pi_l A_l^2 + B + Rb_t^2 - b_{t+1}^2 \quad t \in [T_r, T] \end{aligned}$$

For $t \in [0, T_a)$,

$$\begin{aligned} 0 \leq \sigma c_t^1 + (1 - \sigma)c_t^2 &\leq \sigma(we_t + Rb_t^1 + B - b_{t+1}^1) + (1 - \sigma)(we_t + Rb_t^2 + B - b_{t+1}^2) \\ 0 \leq c_t^\sigma &\leq we_t + Rb_t^\sigma + B - b_{t+1}^\sigma. \end{aligned}$$

Likewise

$$\begin{aligned} 0 \leq c_{T_a}^\sigma &\leq we_{T_a} + Rb_{T_a}^\sigma + B - b_{T_r}^\sigma - A_l^\sigma \\ 0 \leq c_t^\sigma &\leq \pi_l A_l^\sigma + B + Rb_t^\sigma - b_{t+1}^\sigma \quad t \in [T_r, T] \\ b_t^\sigma &= \sigma b_t^1 + (1 - \sigma)b_t^2 \geq 0 \\ A_l^\sigma &= \sigma A_l^1 + (1 - \sigma)A_l^2 \geq 0 \\ b_0^\sigma &= b_T^\sigma = 0. \end{aligned}$$

Thus $\left(\{c_t^\sigma\}_{t=0}^T, \{b_{t+1}^\sigma\}_{t=0}^T, A_l^\sigma\right) \in \Sigma$, and the set of feasible paths is convex. Since the objective function is strictly concave, there will be a unique solution to the first-order conditions. Moreover, $V(A_l)$ is strictly concave. For let $A_l^1, A_l^2 \geq 0$. Then there exists $\left(\{c_t^1\}_{t=0}^T, \{b_{t+1}^1\}_{t=0}^T\right)$ and $\left(\{c_t^2\}_{t=0}^T, \{b_{t+1}^2\}_{t=0}^T\right)$ such that $\left(\{c_t^1\}_{t=0}^T, \{b_{t+1}^1\}_{t=0}^T, A_l^1\right), \left(\{c_t^2\}_{t=0}^T, \{b_{t+1}^2\}_{t=0}^T, A_l^2\right) \in \Sigma$ and

$$V(A_l^1) = \sum_{t=0}^T Q_t \beta^t u(c_t^1)$$

$$V(A_l^2) = \sum_{t=0}^T Q_t \beta^t u(c_t^2).$$

Let $\sigma \in (0, 1)$ and $A_l^\sigma = \sigma A_l^1 + (1 - \sigma)A_l^2$. Then define c_t^σ and b_t^σ by (89)-(90). Since Σ is convex, $(\{c_t^\sigma\}_{t=0}^T, \{b_{t+1}^\sigma\}_{t=0}^T, A_l^\sigma) \in \Sigma$. Let $(\{c_t^*\}_{t=0}^T, \{b_{t+1}^*\}_{t=0}^T)$ be such that $(\{c_t^*\}_{t=0}^T, \{b_{t+1}^*\}_{t=0}^T, A_l^\sigma) \in \Sigma$ and

$$\begin{aligned} V(A_l^\sigma) &= \sum_{t=0}^T Q_t \beta^t u(c_t^*) \geq \sum_{t=0}^T Q_t \beta^t u(c_t^\sigma) \\ &> \sum_{t=0}^T Q_t \beta^t [\sigma u(c_t^1) + (1 - \sigma)u(c_t^2)] \\ &= \sigma V(A_l^1) + (1 - \sigma)V(A_l^2), \end{aligned}$$

where the first inequality follows from the fact that $(\{c_t^\sigma\}_{t=0}^T, \{b_{t+1}^\sigma\}_{t=0}^T, A_l^\sigma)$, $(\{c_t^*\}_{t=0}^T, \{b_{t+1}^*\}_{t=0}^T, A_l^\sigma)$ are both in Σ , but $(\{c_t^*\}_{t=0}^T, \{b_{t+1}^*\}_{t=0}^T)$ are optimal given A_l^σ ; and the second inequality follows from the strict concavity of u .

We cannot solve the model by iteration because, once the equations of motion do not determine the path of c_t across a period when the borrowing constraint binds.

Assume now that u is CRRA with elasticity of intertemporal substitution $\gamma^{-1} > 0$. Let us define the income y_t by

$$y_t = \begin{cases} we_t + B & 0 \leq t < T_a \\ we_t + B - A_l & t = T_a \\ \pi A_l + B & T_a < t \leq T \end{cases}. \quad (92)$$

Then the budget constraint for all t is

$$c_t + b_{t+1} = x_t = y_t + Rb_t \quad (93)$$

where x_t is cash on hand as defined by Deaton (1991). Let us define x_t^0 to be the minimum possible cash on hand. For $t > T_a$, $x_t^0 = y_t$ since the household cannot have negative financial wealth. For $t \leq T_a$, the lifetime budget constraint is

$$\begin{aligned} \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} &= \sum_{s=t}^{T_a} \frac{y_s + Rb_s - b_{s+1}}{R^{s-t}} = \sum_{s=t}^{T_a} \frac{y_s}{R^{s-t}} + Rb_t + R \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t}} - \sum_{s=t}^{T_a-1} \frac{b_{s+1}}{R^{s-t}} - \frac{b_{T_r}}{R^{T_a-t}} \\ \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} + \frac{b_{T_r}}{R^{T_a-t}} &= x_t + \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} + R \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t}} - \sum_{s=t}^{T_a-1} \frac{b_{s+1}}{R^{s-t}} \\ \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} + \frac{b_{T_r}}{R^{T_a-t}} &= x_t + \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} + R \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t}} - \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t-1}} \\ \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} + \frac{b_{T_r}}{R^{T_a-t}} &= x_t + \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} + R \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t}} - R \sum_{s=t+1}^{T_a} \frac{b_s}{R^{s-t}} \\ \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} + \frac{b_{T_r}}{R^{T_a-t}} &= x_t + \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} \end{aligned}$$

Thus for $t \leq T_a$,

$$x_t^0 = - \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} \quad (94)$$

to be the minimum possible cash on hand that allows nonnegative consumptions c_s for $s \geq t$ since the lifetime budget constraint at t is

$$0 \leq \sum_{s=t}^{T_a} \frac{c_s}{R^{s-t}} + \frac{b_{T_r}}{R^{T_a-t}} = x_t + \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} = x_t - x_t^0.$$

If $t < T_a$,

$$\begin{aligned} x_t^0 &= - \sum_{s=t+1}^{T_a} \frac{y_s}{R^{s-t}} = -\frac{y_{t+1}}{R} - \sum_{s=t+2}^{T_a} \frac{y_s}{R^{s-t}} \\ &= -\frac{y_{t+1}}{R} - \frac{1}{R} \sum_{s=t+2}^{T_a} \frac{y_s}{R^{s-t-1}} \\ &= \frac{-y_{t+1} + x_{t+1}^0}{R} \end{aligned}$$

Let us define the terminal value function for $x_T \geq x_T^0 = y_T$,

$$v_T(x_T) = u(x_T). \quad (95)$$

For $T_a \leq t < T$, the Bellman equation for $x_t \geq x_t^0$ is

$$v_t(x_t) = \max_{0 \leq c_t \leq x_t} u(c_t) + \beta \frac{Q_{t+1}}{Q_t} v(y_{t+1} + R(x_t - c_t)). \quad (96)$$

For $0 \leq t < T_a$, the Bellman equation for $x_t \geq x_t^0$ is simply

$$v_t(x_t) = \max_{0 \leq c_t} u(c_t) + \beta \frac{Q_{t+1}}{Q_t} v(y_{t+1} + R(x_t - c_t)). \quad (97)$$

The Lagrangian is

$$\begin{aligned} L_t &= u(c_t) + \beta \frac{Q_{t+1}}{Q_t} v(y_{t+1} + R(x_t - c_t)). \\ \frac{\partial L_t}{\partial c_t} &= u'(c_t) - \beta R \frac{Q_{t+1}}{Q_t} v'(y_{t+1} + R(x_t - c_t)) = 0. \end{aligned}$$

By the Envelope Theorem,

$$v'(x_t) = \frac{\partial L_t}{\partial x_t} = \beta R \frac{Q_{t+1}}{Q_t} v'(y_{t+1} + R(x_t - c_t)) = u'(c_t).$$

If the borrowing constraint does not bind,

$$u'(c_t) = \beta R \frac{Q_{t+1}}{Q_t} u'(c_{t+1}). \quad (98)$$

Proposition 1 For t such that $T_a \leq t \leq T$, there exist $x_t^0 \leq x_t^1 \leq x_t^2 \leq \dots \leq x_t^{T-t+1} = \infty$ such that the solution to (96) for $x_t \in [x_t^i, x_t^{i+1})$ where $i = 0, \dots, T-t$ has the form

$$v_t(x_t) = D_t^i u(x_t + H_t^i) + G_t^i \quad (99)$$

where

$$(D_t^{i+1})^{\frac{1}{\gamma}} = 1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{1}{\gamma}} (D_{t+1}^i)^{\frac{1}{\gamma}} \quad (100)$$

$$H_t^{i+1} = \frac{y_{t+1} + H_{t+1}^i}{R} \quad (101)$$

for $i = 1, \dots, T-t$, and $H_t^0 = 0$ and $D_t^0 = 1$. The optimal consumption rule is

$$c_t(x_t) = (D_t^i)^{-1/\gamma} (x_t + H_t^i). \quad (102)$$

The index i corresponds to the number of periods into the future in which the household will either be dead with certainty or borrowing constrained. For $t = T$, $D_T^0 = 1$, $G_T^0 = 0$, and $H_T^0 = 0$ with

$$c_T = x_T.$$

Now suppose that $T_a \leq t < T$ and suppose that the Proposition holds for $t+1$. Let $x_t \geq x_t^0$. Since v_{t+1} is continuous and strictly concave, by the Theorem of the Maximum, there will be a unique $c_t(x_t)$ such that

$$v_t(x_t) = u(c_t(x_t)) + \beta \frac{Q_{t+1}}{Q_t} v(y_{t+1} + R(x_t - c_t(x_t))),$$

and $c_t(x_t)$ is a continuous function of x_t . Likewise, there will be a unique $\tilde{c}_t(x_t)$ that solves

$$\max_{0 \leq c_t} u(c_t) + \beta \frac{Q_{t+1}}{Q_t} v(y_{t+1} + R(x_t - c_t)),$$

and $\tilde{c}_t(x_t)$ is also continuous.

Suppose the optimal choice \tilde{c}_t is such that $x_{t+1} = y_{t+1} + R(x_t - \tilde{c}_t) \in [x_{t+1}^i, x_{t+1}^{i+1})$ for $i = 0, \dots, T-t-1$. Then

$$\begin{aligned} u'(\tilde{c}_t) &= \beta R \frac{Q_{t+1}}{Q_t} v'(y_{t+1} + R(x_t - \tilde{c}_t)) \\ \tilde{c}_t^{-\gamma} &= \beta R \frac{Q_{t+1}}{Q_t} D_{t+1}^i [y_{t+1} + R(x_t - \tilde{c}_t) + H_{t+1}^i]^{-\gamma} \\ \tilde{c}_t &= \left(\beta R \frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} [y_{t+1} + R(x_t - \tilde{c}_t) + H_{t+1}^i] \end{aligned} \quad (103)$$

Define

$$\phi = (\beta R^{1-\gamma})^{-1/\gamma}. \quad (104)$$

We can rewrite (103) as

$$\begin{aligned} \tilde{c}_t &= \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} - \tilde{c}_t \right] \\ \left[1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right] \tilde{c}_t &= \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \end{aligned}$$

Using (101), we have

$$\tilde{c}_t(x_t) = \frac{\phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}}{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \quad (105)$$

Then

$$\begin{aligned} x_{t+1} &= y_{t+1} + R(x_t - \tilde{c}_t) = R\left(x_t + \frac{y_{t+1}}{R} - \tilde{c}_t\right) \\ &= R\left[x_t + \frac{y_{t+1}}{R} - \frac{\phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}}{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right] \\ &= \frac{R}{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1}}{R} - \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \right]. \end{aligned}$$

For $i = 1, \dots, T-t$, let us define \tilde{x}_t^{i+1} such that $x_{t+1}^i = y_{t+1} + R(\tilde{x}_t^{i+1} + \tilde{c}_t(\tilde{x}_t^{i+1}))$.

$$\begin{aligned} x_{t+1}^i &= \frac{R}{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}} \left[\tilde{x}_t^{i+1} + \frac{y_{t+1}}{R} - \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \right] \\ \frac{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}}{R} x_{t+1}^i &= \tilde{x}_t^{i+1} + \frac{y_{t+1}}{R} - \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \end{aligned}$$

Then

$$\tilde{x}_t^{i+1} = \frac{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}}{R} x_{t+1}^i - \frac{y_{t+1}}{R} + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R}. \quad (106)$$

Since $\tilde{c}_t(x_t)$ is continuous,

$$\lim_{x_t \uparrow \tilde{x}_t^{i+2}} [y_{t+1} + R(x_t + \tilde{c}_t(x_t))] = y_{t+1} + R(\tilde{x}_t^{i+2} + \tilde{c}_t(\tilde{x}_t^{i+2})) = x_{t+1}^{i+1}.$$

Therefore,

$$\frac{R}{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}} \left[\tilde{x}_t^{i+2} + \frac{y_{t+1}}{R} - \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \right] = x_{t+1}^{i+1},$$

and we also have

$$\tilde{x}_t^{i+2} = \frac{1 + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}}}{R} x_{t+1}^{i+1} - \frac{y_{t+1}}{R} + \phi\left(\frac{Q_{t+1}}{Q_t}D_{t+1}^i\right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R}$$

Since $\phi, Q_t, Q_{t+1}, D_{t+1}^i$, and $R > 0$, if $x_{t+1}^i < x_{t+1}^{i+1}$ then $\tilde{x}_t^{i+1} < \tilde{x}_t^{i+2}$. If $x_{t+1}^{i+1} = x_{t+1}^i$, $\tilde{x}_t^{i+2} = \tilde{x}_t^{i+1}$.

Next let us define x_t^1 such that

$$(x_t^1)^{-\gamma} = \beta R \left(\frac{Q_{t+1}}{Q_t} \right) (c_{t+1}(y_{t+1}))^{-\gamma}$$

$$x_t^1 = \left(\beta R \left(\frac{Q_{t+1}}{Q_t} \right) \right)^{-\frac{1}{\gamma}} c_{t+1}(y_{t+1}). \quad (107)$$

Define

$$c_t(x_t) = \begin{cases} \tilde{c}_t(x_t) & x_t > x_t^1 \\ x_t & x_t \leq x_t^1 \end{cases}. \quad (108)$$

Then for $x_t < x_t^1$, $c_t = x_t$ and $x_{t+1} = y_{t+1}$.

$$c_t^{-\gamma} = x_t^{-\gamma} \geq (x_t^1)^{-\gamma} = \beta R \left(\frac{Q_{t+1}}{Q_t} \right) (c_{t+1}(y_{t+1}))^{-\gamma}.$$

For $i = 2, \dots, T-t$,

$$x_t^i = \begin{cases} \tilde{x}_t^i & \tilde{x}_t^i > x_t^1 \\ x_t^1 & \tilde{x}_t^i \leq x_t^1 \end{cases}. \quad (109)$$

Then we still have

$$x_t^0 \leq x_t^1 \leq \dots \leq x_t^{T-t} < x_t^{T-t+1} = \infty.$$

For $x_t \geq x_t^1$, $x_t \in [x_t^i, x_t^{i+1})$ for some $i \in \{1, \dots, T-t\}$.

$$\begin{aligned} c_t(x_t) &= c_t(x_t^1) + \sum_{j=1}^{i-1} \int_{x_t^j}^{x_t^{j+1}} c_t'(x) dx + \int_{x_t^i}^{x_t} c_t'(x) dx \\ &= c_t(x_t^1) + \sum_{j=1}^{i-1} \int_{x_t^j}^{x_t^{j+1}} \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^j \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^j \right)^{-\frac{1}{\gamma}}} dx + \int_{x_t^i}^{x_t} \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} dx \\ &= c_t(x_t^1) + \sum_{j=1}^{i-1} \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^j \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^j \right)^{-\frac{1}{\gamma}}} (x_t^{j+1} - x_t^j) + \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} (x_t - x_t^i) \\ &< c_t(x_t^1) + \sum_{j=1}^{i-1} (x_t^{j+1} - x_t^j) + x_t - x_t^i = c_t(x_t^1) + x_t - x_t^1 = x_t^1 + x_t - x_t^1 = x_t, \end{aligned}$$

and

$$(c_t(x_t))^{-\gamma} = \beta R \left(\frac{Q_{t+1}}{Q_t} \right)^{-\gamma} (c_{t+1}(y_{t+1} + R(x_t - c_t(x_t))))^{-\gamma}.$$

Suppose that $\gamma = 1$. Then if $x_t \in [x_t^{i+1}, x_t^{i+2})$ for some $i \in \{0, \dots, T-t-1\}$,

$$\begin{aligned} v_t(x_t) &= \ln(c_t(x_t)) + \beta \frac{Q_{t+1}}{Q_t} [D_{t+1}^i \ln(y_{t+1} + R(x_t - c_t(x_t))) + H_{t+1}^i] + G_{t+1}^i \\ &= \ln \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right) \\ &\quad + \beta \frac{Q_{t+1}}{Q_t} \left[D_{t+1}^i \ln \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1}}{R} - \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \right] + H_{t+1}^i \right) + G_{t+1}^i \right] \end{aligned}$$

$$\begin{aligned}
v_t(x_t) &= \ln \left(\frac{\beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right) \\
&\quad + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \ln \left(\frac{R}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i
\end{aligned}$$

Since

$$H_t^{i+1} = \frac{y_{t+1} + H_{t+1}^i}{R},$$

$$\begin{aligned}
v_t(x_t) &= \ln \left(\frac{\beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} [x_t + H_t^{i+1}] \right) \\
&\quad + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \ln \left(\frac{R}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} [x_t + H_t^{i+1}] \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i
\end{aligned}$$

$$\begin{aligned}
v_t(x_t) &= \left(1 + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \right) \ln(x_t + H_t^{i+1}) + \ln \left(\frac{\beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \right) \\
&\quad + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \ln \left(\frac{R}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i
\end{aligned}$$

By (100),

$$D_t^{i+1} = 1 + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i$$

and we can define

$$G_t^{i+1} = \ln \left(\frac{\beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \right) + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \ln \left(\frac{R}{1 + \beta^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-1}} \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i$$

$$G_t^{i+1} = -\ln(D_t^{i+1}) + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \ln \left(\frac{\beta R \frac{Q_{t+1}}{Q_t} D_{t+1}^i}{D_t^{i+1}} \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i$$

$$\beta R \frac{Q_{t+1}}{Q_t} D_{t+1}^i = R(D_t^{i+1} - 1)$$

$$G_t^{i+1} = -\ln(D_t^{i+1}) + (D_t^{i+1} - 1) \ln \left(\frac{R(D_t^{i+1} - 1)}{D_t^{i+1}} \right) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i.$$

For $x_t \in [x_t^0, x_t^1)$,

$$v_t(x_t) = \ln(c_t(x_t)) + \beta \frac{Q_{t+1}}{Q_t} v_{t+1}(y_{t+1}) = \ln(x_t) + \beta \frac{Q_{t+1}}{Q_t} v_{t+1}(y_{t+1}).$$

Thus $D_t^0 = 1$, $H_t^0 = 0$ and

$$G_t^0 = \beta \frac{Q_{t+1}}{Q_t} v_{t+1}(y_t).$$

Suppose that $\gamma \neq 1$. Then if $x_t \in [x_t^{i+1}, x_t^{i+2})$ for some $i \in \{0, \dots, T-t-1\}$,

$$\begin{aligned} v_t(x_t) &= \frac{1}{1-\gamma} (c_t(x_t))^{1-\gamma} + \beta \frac{Q_{t+1}}{Q_t} \left[\frac{D_{t+1}^i}{1-\gamma} (y_{t+1} + R(x_t - c_t(x_t)) + H_{t+1}^i)^{1-\gamma} + G_{t+1}^i \right] \\ &= \frac{1}{1-\gamma} \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right)^{1-\gamma} \\ &\quad + \beta \frac{Q_{t+1}}{Q_t} \left[\frac{D_{t+1}^i}{1-\gamma} \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1}}{R} - \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \right] + H_{t+1}^i \right)^{1-\gamma} + G_{t+1}^i \right] \\ v_t(x_t) &= \frac{1}{1-\gamma} \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right)^{1-\gamma} \\ &\quad + \beta \frac{Q_{t+1}}{Q_t} \left[\frac{D_{t+1}^i}{1-\gamma} \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \left[x_t + \frac{y_{t+1} + H_{t+1}^i}{R} \right] \right)^{1-\gamma} + G_{t+1}^i \right] \end{aligned}$$

Since

$$H_t^{i+1} = \frac{y_{t+1} + H_{t+1}^i}{R},$$

$$\begin{aligned} v_t(x_t) &= \frac{1}{1-\gamma} \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} [x_t + H_t^{i+1}] \right)^{1-\gamma} \\ &\quad + \beta \frac{Q_{t+1}}{Q_t} \frac{D_{t+1}^i}{1-\gamma} \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} [x_t + H_t^{i+1}] \right)^{1-\gamma} + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i \\ v_t(x_t) &= \left[\left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \right)^{1-\gamma} + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \right)^{1-\gamma} \right] u(x_t + H_t^{i+1}) + \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i \end{aligned}$$

Thus

$$D_t^{i+1} = \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \right)^{1-\gamma} + \beta \frac{Q_{t+1}}{Q_t} D_{t+1}^i \left(\frac{R}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \right)^{1-\gamma}$$

$$G_t^{i+1} = \beta \frac{Q_{t+1}}{Q_t} G_{t+1}^i$$

$$D_t^{i+1} = \frac{1}{\left(1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right)^{1-\gamma}} \left[\phi^{1-\gamma} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{\frac{\gamma-1}{\gamma}} + \frac{Q_{t+1}}{Q_t} D_{t+1}^i \phi^{-\gamma} \right]$$

$$= \frac{\frac{Q_{t+1}}{Q_t} D_{t+1}^i \phi^{-\gamma} \left(1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right)}{\left(1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right)^{1-\gamma}}$$

$$= \frac{\frac{Q_{t+1}}{Q_t} D_{t+1}^i \phi^{-\gamma}}{\left(1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right)^{-\gamma}}$$

$$= \left(\frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \right)^{-\gamma}$$

$$D_t^{i+1} = \left(1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{\frac{1}{\gamma}} \right)^{\gamma}$$

$$(D_t^{i+1})^{\frac{1}{\gamma}} = 1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{\frac{1}{\gamma}} (D_{t+1}^i)^{\frac{1}{\gamma}}$$

For $x_t \in [x_t^0, x_t^1)$,

$$v_t(x_t) = \frac{1}{1-\gamma} (c_t(x_t))^{1-\gamma} + \beta \frac{Q_{t+1}}{Q_t} v_{t+1}(y_{t+1}).$$

Thus $D_t^0 = 1$, $H_t^0 = 0$ and

$$G_t^0 = \beta \frac{Q_{t+1}}{Q_t} v_{t+1}(y_t).$$

Note that for $x_t \in [x_t^{i+1}, x_t^{i+2})$ where $i \in \{0, 1, \dots, T-t-1\}$

$$c_t(x_t) = \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} [x_t + H_t^{i+1}]$$

$$= \frac{x_t + H_t^{i+1}}{1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{\frac{1}{\gamma}}}$$

$$= (D_t^{i+1})^{-\frac{1}{\gamma}} (x_t + H_t^{i+1})$$

For $x_t \in [x_t^0, x_t^1)$,

$$c_t(x_t) = x_t = (D_t^0)^{-\frac{1}{\gamma}} (x_t + H_t^0)$$

Proposition 2 For t such that $0 \leq t < T_a$, there exist $x_t^0 = x_t^{T_a-t} \leq x_t^{T_a-t+1} \leq x_t^{T_a+2} \leq \dots \leq x_t^{T-t+1} = \infty$ such that the solution to (96) for $x_t \in [x_t^i, x_t^{i+1})$ where $i = T_a - t, \dots, T - t$ has the form

$$v_t(x_t) = D_t^i u(x_t + H_t^i) + G_t^i$$

where

$$(D_t^{i+1})^{\frac{1}{\gamma}} = 1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{1}{\gamma}} (D_{t+1}^i)^{\frac{1}{\gamma}}$$

and

$$H_t^{i+1} = \frac{y_{t+1} + H_{t+1}^i}{R}.$$

The optimal consumption rule is

$$c_t(x_t) = (D_t^{i+1})^{-1/\gamma} (x_t + H_t^{i+1}).$$

The proof is the same as for Proposition 1 without the borrowing constraint.

Let us define

$$m_t^i = (D_t^i)^{-1/\gamma} \tag{110}$$

to be the MPC at age t on $[x_t^i, x_t^{i+1})$. Then, from (100), the MPC satisfies

$$\begin{aligned} \frac{1}{m_t^{i+1}} &= 1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{1}{\gamma}} \frac{1}{m_{t+1}^i}. \\ m_t^{i+1} &= \frac{m_{t+1}^i}{m_{t+1}^i + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{1}{\gamma}}} \in [0, 1] \end{aligned}$$

if $m_{t+1}^i > 0$, which it will be.

$$\begin{aligned} x_{t+1}^i &= y_{t+1} + R (\tilde{x}_t^{i+1} - c_t(\tilde{x}_t^{i+1})) \\ &= y_{t+1} + R (\tilde{x}_t^{i+1} - m_t^{i+1} (\tilde{x}_t^{i+1} + H_t^{i+1})) \\ &= y_{t+1} + R (1 - m_t^{i+1}) \tilde{x}_t^{i+1} - R m_t^{i+1} H_t^{i+1} \end{aligned}$$

$$\tilde{x}_t^{i+1} = \frac{1}{1 - m_t^{i+1}} \frac{x_{t+1}^i - y_{t+1} + R m_t^{i+1} H_t^{i+1}}{R} \tag{111}$$

$$\begin{aligned} m_t^{i+1} &= \frac{1}{1 + \phi^{-1} \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{\frac{1}{\gamma}}} = \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \\ 1 - m_t^{i+1} &= \frac{1}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} \\ \frac{1}{1 - m_t^{i+1}} &= 1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \end{aligned}$$

$$\begin{aligned}
\frac{m_t^{i+1}}{1 - m_t^{i+1}} &= \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \\
\tilde{x}_t^{i+1} &= \frac{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{R} x_{t+1}^i - \frac{y_{t+1}}{R} + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \frac{H_{t+1}^i}{R} \\
&= \frac{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{R} x_{t+1}^i - \frac{\left[1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right] y_{t+1}}{R} + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \frac{y_{t+1} + H_{t+1}^i}{R} \\
&= \left(1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}} \right) \frac{\left[x_{t+1}^i - y_{t+1} + \frac{\phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}}{1 + \phi \left(\frac{Q_{t+1}}{Q_t} D_{t+1}^i \right)^{-\frac{1}{\gamma}}} H_{t+1}^i \right]}{R} \\
&= \frac{1}{1 - m_t^{i+1}} \frac{x_{t+1}^i - y_{t+1} + m_t^{i+1} H_{t+1}^i}{R}
\end{aligned}$$

The maximum annuity that we can consider is the age- T_a future value of preannuitization income.

4.1 Rational Behavior Without the longevity Annuity

Let us reconsider the problem of a rational household for the case where there is no longevity annuity available in the market. The first-order conditions (77), (78), (79), and (81) still hold. Then (81) implies

$$\lambda_{T_a} \geq \pi_b \sum_{t=T_r}^T \lambda_t$$

with inequality only if $A_b = 0$. If none of the bond borrowing constraints bind, we again have (84).

$$R^{-T_a} \lambda_0 \geq \pi_b \sum_{t=T_r}^T R^{-t} \lambda_0,$$

$$\frac{1}{\sum_{t=T_r}^T R^{-(t-T_a)}} \geq \pi_b,$$

but the left-hand side is π_b , so we can have $A_b = 0$. The internal rate of return on the bond annuity is r , so the household is indifferent between investing in the bond annuity and the bond. Any portfolio that achieves

$$Q_t \beta^t u'(c_t) = Q_0 \beta^t u'(c_0) R^{-t}$$

will give the same utility. We could have a borrowing constraint bind because of the bequest income though, although this is unlikely. That is the only situation where the household might care whether it invests in the bond annuity or not.

A Details of the Model

A.1 Dynamics Underlying the longevity Annuity

The assets for age t will have a solution of the form

$$a_t^l = R^t \chi_l + \zeta_l \sum_{s=T_a+1}^{t-1} R^{t-s} Q_s,$$

where χ_l and ζ_l are constants to be determined.

$$\pi_l A_l Q_t + R^{t+1} \chi_l + \zeta_l \sum_{s=T_a+1}^t R^{t+1-s} Q_s = R^{t+1} \chi_l + \zeta_l R \sum_{s=T_a+1}^{t-1} R^{t-s} Q_s$$

$$\pi_l A_l Q_t + \zeta_l \sum_{s=T_a+1}^t R^{t+1-s} Q_s = \zeta_l \sum_{s=T_a+1}^{t-1} R^{t+1-s} Q_s$$

$$\pi_l A_l Q_t + \zeta_l R Q_t = 0$$

$$\zeta_l = -\frac{\pi_l A_l}{R}$$

$$a_t^l = R^t \chi_l - \pi_l A_l \sum_{s=T_a+1}^{t-1} R^{t-1-s} Q_s$$

To satisfy the initial condition, we need

$$a_{T_a+1}^l = A_l Q_{T_a} = R^{T_a+1} \chi_l - 0$$

$$\chi_l = \frac{A_l Q_{T_a}}{R^{T_a+1}}$$

Thus

$$a_t^l = R^{t-T_a-1} A_l Q_{T_a} - \pi_l A_l \sum_{s=T_a+1}^{t-1} R^{t-1-s} Q_s. \quad (112)$$

To satisfy the condition (9), we must have

$$0 = R^{T-T_a} A_l Q_{T_a} - \pi_l A_l \sum_{s=T_a+1}^T R^{T-s} Q_s$$

$$R^{-T_a} Q_{T_a} = \pi_l \sum_{s=T_a+1}^T R^{-s} Q_s$$

$$\pi_l = \frac{Q_{T_a}}{\sum_{s=T_a+1}^T R^{T_a-s} Q_s}.$$

Substituting this back into (112), we get

$$\begin{aligned}
a_t^l &= R^{t-T_a-1} A_l Q_{T_a} - \frac{A_l}{\sum_{s=T_a+1}^T \frac{Q_{s|T_a}}{R^{s-T_a}}} \sum_{s=T_a+1}^{t-1} R^{t-1-s} Q_s. \\
a_t^l &= A_l R^{t-1} \left[R^{-T_a} Q_{T_a} - \frac{\sum_{s=T_a+1}^{t-1} R^{-s} Q_s}{\sum_{s=T_a+1}^T \frac{Q_{s|T_a}}{R^{s-T_a}}} \right]. \\
a_t^l &= A_l R^{t-1} \left[R^{-T_a} Q_{T_a} - \frac{\sum_{s=T_a+1}^{t-1} R^{-s} Q_s}{\sum_{s=T_a+1}^T \frac{Q_{s|T_a}}{R^{s-T_a}}} \right]. \\
a_t^l &= A_l R^{t-1} \left[R^{-T_a} Q_{T_a} - \frac{Q_{T_a} \sum_{s=T_a+1}^{t-1} R^{-s} Q_s}{R^{T_a} \sum_{s=T_a+1}^T R^{-s} Q_s} \right]. \\
a_t^l &= R^{t-T_a-1} A_l Q_{T_a} \left[1 - \frac{\sum_{s=T_a+1}^{t-1} R^{-s} Q_s}{\sum_{s=T_a+1}^T R^{-s} Q_s} \right].
\end{aligned}$$

Note that

$$\begin{aligned}
a_{T_a+1}^l &= R^{T_a+1-T_a-1} A_l Q_{T_a} \left[1 - \frac{\sum_{s=T_a+1}^{T_a} R^{-s} Q_s}{\sum_{s=T_a+1}^T R^{-s} Q_s} \right] = A_l Q_{T_a} \\
a_{T+1}^l &= R^{T+1-T_a-1} A_l Q_{T_a} \left[1 - \frac{\sum_{s=T_a+1}^T R^{-s} Q_s}{\sum_{s=T_a+1}^T R^{-s} Q_s} \right] = 0.
\end{aligned}$$

A.2 Dynamics of the Bond Annuity

For $t = T_a + 1, \dots, T + 1$,

$$a_t^b = \chi_b R^t + \zeta_b$$

where χ_b and ζ_b are constants to be determined.

$$\begin{aligned}
\chi_b R^{t+1} + \zeta_b &= R(\chi_b R^t + \zeta_b) - \pi_b A_b \\
\zeta_b &= R\zeta_b - \pi_b A_b
\end{aligned}$$

If we let $r = R - 1$ be the net return on saving, this becomes

$$\begin{aligned}
r\zeta_b &= \pi_b A_b \\
\zeta_b &= \frac{\pi_b A_b}{r}
\end{aligned}$$

The initial condition then gives

$$\begin{aligned}
A_b &= \chi_b R^{T_r} + \frac{\pi_b A_b}{r} \\
\chi_b R^{T_r} &= \left(1 - \frac{\pi_b}{r}\right) A_b \\
\chi_b &= R^{-T_r} \left(1 - \frac{\pi_b}{r}\right) A_b
\end{aligned}$$

Thus the assets underlying the bond annuity at age t will be

$$a_t^b = R^{t-T_r} \left(1 - \frac{\pi_b}{r}\right) A_b + \frac{\pi_b}{r} A_b$$

The terminal condition then implies

$$\begin{aligned} a_{T+1}^b &= R^{T+1-T_r} \left(1 - \frac{\pi_b}{r}\right) A_b + \frac{\pi_b}{r} A_b = 0 \\ 0 &= R^{T-T_a} \left(1 - \frac{\pi_b}{r}\right) + \frac{\pi_b}{r} \end{aligned}$$

$$\begin{aligned} R^{T-T_a} &= (R^{T-T_a} - 1) \frac{\pi_b}{r} \\ \pi_b &= \frac{r R^{T-T_a}}{R^{T-T_a} - 1} \end{aligned}$$

Thus the payout on the bond annuity is

$$\pi_b = \frac{r}{1 - R^{-(T-T_a)}}$$

This can also be written

$$\begin{aligned} \pi_b &= \left(\frac{1 - R^{-(T-T_a)}}{R - 1} \right)^{-1} \\ &= \left(\frac{1}{R} \frac{1 - R^{-(T-T_a)}}{1 - R^{-1}} \right)^{-1} \\ &= \left(\frac{1}{R} \sum_{t=0}^{T-T_a-1} \frac{1}{R^t} \right)^{-1} \\ &= \left(\sum_{t=1}^{T-T_a} \frac{1}{R^t} \right)^{-1} \\ \pi_b &= \left(\sum_{t=T_r}^T \frac{1}{R^{t-T_a}} \right)^{-1} \end{aligned} \tag{113}$$

A.3 Verifying the Income-Expenditure Identity

Aggregate consumption is

$$\begin{aligned}
C &= \sum_{t=0}^T Q_t c_t \\
&= \sum_{t=0}^{T_a-1} Q_t c_t + Q_{T_a} c_{T_a} + \sum_{t=T_a+1}^T Q_t c_t \\
&= \sum_{t=0}^{T_a-1} Q_t [w e_t + R b_t + B - b_{t+1}] + Q_{T_a} [w e_{T_a} + R b_{T_a} + B - A_l - A_b - b_{t+1}] \\
&\quad + \sum_{t=T_a+1}^T Q_t [\pi_b A_b + \pi_l A_l + R b_t + B - b_{t+1}] \\
&= w \sum_{t=0}^{T_a} Q_t e_t + R \sum_{t=0}^T Q_t b_t + B \sum_{t=0}^T Q_t - \sum_{t=0}^T Q_t b_{t+1} - (A_l + A_b) Q_{T_a} + (\pi_l A_l + \pi_b A_b) \sum_{t=T_a+1}^T Q_t \\
&= wN + R \sum_{t=0}^T Q_t b_t + R \sum_{t=0}^T (Q_t - Q_{t+1}) b_{t+1} + R \sum_{t=T_a}^T (Q_t - Q_{t+1}) a_{t+1}^b \\
&\quad - K_b - a_{T_a+1}^l - Q_{T_a} a_{T_a+1}^b + \sum_{t=T_a+1}^T [R a_t^l - a_{t+1}^l + Q_t (R a_t^b - a_{t+1}^b)] \\
&= wN + R \sum_{t=0}^T Q_t b_t + (R-1)K_b + R K_a^b - R \sum_{t=1}^{T+1} Q_t b_t - R \sum_{t=T_a+1}^{T+1} Q_t a_t^b \\
&\quad - \sum_{t=T_a}^T a_{t+1} + R \sum_{t=T_a+1}^T a_t - \sum_{t=T_a}^T Q_t a_{t+1}^b + R \sum_{t=T_a+1}^T Q_t a_t^b \\
&= wN + R \sum_{t=0}^T Q_t b_t + (R-1)K_b + R K_a^b - R \sum_{t=0}^T Q_t b_t - R \sum_{t=T_a+1}^T Q_t \\
&\quad - a_t^b - \sum_{t=T_a}^T a_{t+1} + R \sum_{t=T_a+1}^{T+1} a_t - K_a^b + R \sum_{t=T_a+1}^T Q_t a_t^b \\
&= wN + (R-1)K_b + (R-1)K_a^b - \sum_{t=T_a}^T a_{t+1} + R \sum_{t=T_a}^T a_{t+1} \\
&= wN + (R-1)(K_b + K_a^b + K_a) \\
&= wN + (R-1)K \\
&= wN + rK \\
&= Y - \delta K
\end{aligned}$$

Thus the income-expenditure identity holds.

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