Temptation and Subjective Feasibility

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Abstract: This paper develops a new model of menu choice induced by temptation. The presence of temptation creates a utility cost in that choices cannot be recovered by maximization of consumption utility. However, in our model the cost of temptation is borne through constraints on the DM's choice domain as opposed to an explicit cost function. The idea is that, within menu, objects are grouped together by attribute and the DM is only able to select an attribute possessed by a target choice, as opposed to selecting the choice itself. For example, when the DM looks at a dinner menu he can partly control himself by looking at the low-calorie portion of the menu, but within this subset he might be drawn to the most tempting option. The main feature of our model is the manner in which choices are grouped by attributes, we refer to these as *categories*. Categories are subjective, so that the exercise of the paper is to show how these can be elicited and identified from choices over menus.

Keywords: Menu Choice, Preference for Commitment, Temptation.

1 Introduction

This paper revisits the issue of temptation-driven preferences. We begin by recalling the choice environment. In period 1 (the menu choice stage), a decision-maker (DM) commits to some option set (menu) of period 2 choices. In period 2 (the consumption choice stage), the DM makes a selection from the period 1 menu. Choices in period 2 might look anomalous since they need not agree with the period 1 ranking over consumption commitments, i.e. singleton menus. The latter ranking is often labelled the "normative" preference. In cases where the period 2 choices do not agree with the maximizer of normative preference we will say that the DM is constrained by "temptation". For a typical example, imagine the DM expresses a period 1 preference for a hard deadline to complete a project, e.g. he chooses the option of a hard deadline over a flexible deadline whenever both are offered. This indicates a (normative) preference not to delay. On the other hand, we also observe that the DM chooses to delay completion when he is not offered a firm deadline. One explanation for examples of this kind is that there is some unobservable constraint (temptation) affecting the DM at the point of choice that prevents him/her from choosing the normative maximizer on the menu.

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The decision-theory literature on temptation, after Gul and Pesendorfer (2001) (GP), provides a time-consistent explanation for this conflict between normative preference and period 2 choice. The GP model shows that we can recover this seemingly inconsistent behavior via maximization of an alternative "net welfare" ranking on choices. This ranking is composed from two utility functions u, v representing (resp.) normative and "temptation" preference. The net welfare of a choice is the difference between its normative value and the cost of choosing this option over the v-maximizer (i.e. the most tempting options) on the menu. Choices which maximize net welfare fall into one of two categories. Either the DM pays a utility cost to select something other than a v-maximizer on the menu or the DM is selecting the v-maximizer on the menu. In the former case we say that the DM is exerting (costly) self-control and in the latter that he is giving in to temptation. On account of this dichotomy, we refer to this approach to modeling temptation as the "costly self-control" approach.

In this paper, we develop a different model of temptation. The motivation comes from a class of examples where the DM is typically giving in to temptation, but asserts self-control in how severely he gives in. Casual empiricism suggests that there are settings in which this describes how temptation distorts choices: (i) a DM tempted by multiple gambling options mitigates this temptation by choosing low stakes options (e.g. slot machines), (ii) a flexible completion deadline induces some procrastination but not indefinite delay, (iii) deviations from a diet involve a minor indulgence which only occur in the presence of more severe temptations, and so on. In each of these cases, background temptations distort the DM's choice even though the selected choice is not by itself particularly tempting – so that the DM can avoid this option when it is offered in isolation. In this sense, we think of the DM's choice as a proxy commitment device and the act of self-control is in avoiding the more harmful temptations on the menu by virtue of giving in to the less harmful ones. This type of temptation-induced choice is important. First, in addition to its plausibility, there is some experimental evidence for this behavior as it nests (temptation-driven) versions of two well-documented choice anomalies, viz. the compromise and attraction effects. Second, DM's who respond to temptation in this manner offer more room for welfare-improving intervention than DM's with costly self-control preferences (after Gul and Pesendorfer (2001)).

The latter property is a consequence of two features. First, the menu of temptations induces a choice that the DM would not be tempted enough to choose otherwise. Second, choice distortion takes the form of a nudge rather than a push, e.g. playing the slot machines rather than a high-stakes card game. In these situations, a policy which prohibits tempting options (usually) improves welfare since it removes those options which are inducing the nudge. Recall that an important feature of the GP model is that the DM only expresses a preference for commitment when he is *not*

giving in to temptation. Hence, in the GP model partially prohibitive policies which take away some options can harm DM's who are giving in to temptation because this was less costly than exerting self-control, which intervention now forces upon them.[1] The reason for the different welfare implications is that, for the behavior considered here, changes in welfare can only occur by changing choices. This type of choice behavior is, in general, difficult to capture using a costly self-control approach, after Gul and Pesendorfer (2001).[2] At the same time, temptation must create costs somehow since the DM expresses a preference for commitment. Moreover, the DM is exerting self-control since he is just allowing himself to be nudged by temptation. Thus, we need to find an alternative way to model both the welfare cost of temptation and the fact that the DM is optimally exerting self-control against temptation.

The idea behind our model is that choices are implicit bundles of attributes and these attributes can account for the choice of one object over another. Observing a choice of x over y (more specifically, a menu containing x over a menu containing y) we should be able to explain this choice as arising from a choice of an attribute possessed by x over attribute(s) possessed by y. Accordingly, we recast the DM's optimization problem over the proxy choice domain of attributes. The decision procedure is as follows. In the menu choice phase, the DM evaluates a menu by grouping consumption choices by attributes. If two consumption choices differ in some attribute, then the DM can use that attribute as a commitment device to select one choice over another. We view the (implicit) choice of an attribute as a period 1 decision, i.e. when the DM selects a given menu it is as if he commits to only selecting a consumption choice with a certain attribute. In this sense, the set of attributes is the self-control variable in the model. When these map one-to-one to actual choices, the DM has perfect self-control and we recover the standard model. However, in the given menu there may be more than one consumption choice with the chosen attribute. This indeterminacy is resolved in period 2 in which the DM, having "chosen" an attribute and anticipating temptation's influence, realizes that he may not be able to stay away from tempting options which possess that attribute. Hence, with a worst-case evaluation in mind, the period 1 assessment of a menu is the value of the least-worst temptation, i.e. equivalently the DM chooses the attribute possessed by the least harmful temptation.

To illustrate the procedure, imagine the DM values timely completion of a project but is tempted to shirk when given a flexible deadline. A consumption choice here

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[1]The analysis in Gul and Pesendorfer (2007) shows that price policies (e.g. taxing temptations) can affect choices but can also distort welfare downwards. From this perspective, prohibitive policies are better than taxes since they can (so long as the DM is not giving in) lead to a welfare increase, even though they might not affect choices.

[2]The source of the difficulty is a subtle point that requires more details to do it full justice. We refer the interested reader to section 4.

would be a completion time. The DM can control his desire to shirk by grouping completion times into categories, e.g. a little late, late, very late, etc. In this way he can partially control the temptation to delay. However, these categories do not perfectly discriminate between choices – so that the decision maker cannot separate two choices which offer similar levels of delay. Hence, all else equal, given two completion times that fall into the same group the temptation to shirk wins over and the DM selects – within that group – the completion time which affords maximal delay. That the "level of delay" variable only coarsely maps to choices and that this is the only attribute that he can use to discriminate between choices is how temptation creates welfare distortion, viz. the presence of a flexible deadline nudges a DM, who has a normative preference for timely completion, into a choice of incremental delay. More generally, there may be multiple attributes that the DM can use to discriminate among choices. Each attribute serves as a commitment device, but – as in the previous example – there is a limit to how well this controls period 2 choice on account of multiple choices which possess this attribute.

An attribute in our model is a subset of choices consisting of those elements in consumption space (implicitly) possessing that attribute. Moreover, the space of attributes is a subjective construct. After Gul et al. (2010), the intuition is that it is not so much the physical attributes of the choices that matter – these may well be observable, so that we can just enlarge the objective choice space to take objective attribute descriptions into account. What matters are those physical attributes that affect the DM's choice. Attributes and attribute combinations relevant for choice are subjective objects and must be elicited from choice behavior. The main results of the paper elicit and identify attributes taking preferences over menus as primitive. The menu choice viewpoint is important since our interpretation of the model, viz. where the DM anticipates temptation and solves an optimization to assess the ex ante value of a menu, makes more sense when there is a temporal lag between the point of commitment (when the DM chooses an attribute) and the point of consumption (when he yields to temptation). On the other hand, temptation causes welfare loss only by changing choices (in our model). Hence, it is natural to ask whether we can turn the table around on the typical menu preference exercise and, assuming period 2 choice, try to recover the implicit model over menus that generated this period 2 choice.

While welfare loss only occurs via choice distortion in our model, a critical feature is that there is non-trivial preference for commitment. For example, the menu preference exhibits non-degenerate Set-Betweenness (after Gul and Pesendorfer (2001)). Nevertheless, it turns out that the model can be recovered from an *a priori* weaker set of observables, viz. if we observe not just period 2 choices but also the normative ranking, i.e. the DM's ranking on singleton menus. These two observables turn out to be sufficient to recover the model over menus. Having mentioned this equivalence,

however, we believe the most intuitive way to view the model is as a particular example of optimal self-control in anticipation of temptation (as described above). Hence, when we initially introduce primitives and axioms in the next section we use the language of menu preferences. Section 3 presents our two main results – a representation theorem for the model and identification of the space of attributes. In section 4 we compare with other temptation models and present the results recovering the model from the ex post observables. All proofs are in the appendix.

1.1 Related Literature

This paper is most closely related to the axiomatic literature on temptation and self-control. Our menu preferences are "temptation-driven" in the spirit of Dekel et al. (2009) (DLR). The model is inspired by some of the examples in DLR, which are used to motivate a multiple temptation generalization of Gul and Pesendorfer (2001). The DLR temptation model is still one of costly self-control, hence it cannot explain the types of temptation we are interested in modeling. However, we should add that our examples are themselves modifications of the DLR examples. The model in Noor and Takeoka (2010) can explain some of our examples (viz. the compromise effect), but not others (viz. aggregation effects). The connections to the Gul and Pesendorfer (2001), Dekel et al. (2009), and Noor and Takeoka (2010) models are examined in section 4.

There is also a "reduced form" representation of the model where the utility on menus can be expressed as maximization of the normative preference on a choice filter, following the recent literature on quasi-rational choice, e.g. Lleras et al. (2008), Ok et al. (2010), de Clippel and Eliaz (2012), Manzini and Mariotti (2012), Masatlioglu et al. (2012), Cherepanov et al. (2013). This group of papers develops models of two-stage constrained optimization which can also be expressed as maximization of a preference against a choice filter.[3] In section 4, we characterize conditions on the choice filter that determine when it is derived from the (ex ante) model over menus.

2 Model

The choice environment is described as follows.

- Let $X = \{x_1, \ldots, x_n\}$ be an enumeration of the consumption space.

- Let $\mathcal{M}$ denote the collection of non-empty subsets of X (menus).

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[3]For some of the papers the models explicitly invoke choice filters as a parameter and for others the choice filter is a reduced form representation.

- Let $\mathbf{P}(X)$ be the set of complete, transitive preference relations defined on $\mathcal{M}$.

The behavioral primitive for this paper is a preference relation $\succeq \, \in \mathbf{P}(X)$. Henceforth, preference relations will always be assumed to be complete and transitive. For any menu A we take $\sup(A), \inf(A)$ to be (resp.) the set of $(\succeq)|_X$-maximal (resp. $(\succeq)|_X$-minimal) elements in the menu A, where $(\succeq)|_X$ is the ranking on singleton menus. Finally, we denote $(-\infty, x)$ (resp. $(x, \infty), [x, \infty), (-\infty, x]$, and so on) to be the $(\succeq)|_X$-order interval, $(-\infty, x) := \{y \in X : \{x\} \succ \{y\}\}$.

2.1 Model Description

The model has two parameters: (i) a collection of subsets of the consumption space, denoted $\mathcal{C} \equiv \{\mathcal{C}_i\}_i$, with the sole restriction that $\cup_i \mathcal{C}_i = X$ and (ii) a cardinal representation, $u(\cdot)$, of the ranking on consumption choices. These are put together to form the following utility on menus:

$$U(A) := \max_{\mathcal{C}_i : \mathcal{C}_i \cap A \neq \emptyset} \min_{x \in \mathcal{C}_i \cap A} u(x)$$

We call this utility on menus the *category model*, denoted in short via the pair $(u, \mathcal{C})$. Hereafter, we omit the requirement that the maximum is taken only over those i such that $\mathcal{C}_i \cap A \neq \emptyset$.[4] The model is summarized in two steps. In step 1 the DM anticipates temptation and optimally chooses an attribute. This means that he commits to avoid choices which don't posses this attribute, e.g. he commits to avoid high risk gambles, calorie rich desserts, excessive delay opportunities, and so on. In step 2, within the set of consumption choices possessing the chosen attribute, the DM gives in to temptation and picks the most tempting alternative.

There are many ways of modeling how the DM succumbs to temptation. For instance, a milder form of succumbing to temptation might have the DM yielding to temptation by maximizing some v (à la Strotz), or milder yet, a collection of v_i's (where the selected v_i is allowed to depend on the menu).[5] These are more general ways of giving in to temptation, e.g. than just taking $v = -u$, were there no restriction on the domain of these functions. However, we evaluate the ex post choice function not on the full choice space, but on the subset of choices which possess a selected attribute – where attributes are themselves choice variables. Hence, with an appropriately enlarged attribute space the manner in which we have described the DM as giving in to temptation, i.e. minimizing u on a chosen attribute domain, is rich enough to nest all of the listed alternatives.[6] The max-min structure does,

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[4]This can be avoided if we declare, by fiat, the minimum of a function over the empty-set to be some $-K$ where $-K < u(x), \forall x$.

[5]By this we mean a DM who gives in to temptation via $\arg\max_{x \in \cup_i A_{v_i}} u(x)$, where $A_{v_i} = \arg\max_{y \in A} v_i(y)$.

[6]The manner in which the nesting occurs involves some details and is delayed to section 4 (see footnote 14), wherein we compare with the Strotz model and costly self-control models.

however, assume an ex ante attitude towards the impact of temptation – since the DM uses a worst-case evaluation to assess menus. In section 4 we find the source of this attitude towards temptation.

Recall the GP model: $U(A) = \max_{x \in A} [u(x) - [\max_{y \in A} v(y) - v(x))]$. In this model, the optimal choice involves a tradeoff with consumption utility, $u(x)$, and self-control cost, $\max_{y \in A} v(y) - v(x)$. In the category model, the (anticipated) ex post choice can also be expressed as an outcome of tradeoff. To see this, it is useful to rewrite the model as follows. For each $x \in X$, put

$$\mathcal{C}(x) := \{\mathcal{C}_i \in \mathcal{C} : x \in \sup(\mathcal{C}_i)\}$$

That is, $\mathcal{C}(x)$ is the (possibly empty) sub-collection of $\mathcal{C}$ consisting of sets whose u-maximal element is x. Note that we have $\mathcal{C} = \cup_{x \in X} \mathcal{C}(x)$. We can rewrite the category model:

$$U(A) = \max_{\mathcal{C}_i \in \mathcal{C}} \min_{z \in \mathcal{C}_i \cap A} u(z)$$
$$= \max_{x \in X} [\max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{z \in \mathcal{C}_i \cap A} u(z)]$$

The full maximization problem breaks up into a collection of smaller problems where, in each subproblem, the DM's choice domain is the collection $\mathcal{C}(x)$. Notice that x is the unconstrained welfare optimum over all elements in the sets $\mathcal{C}_i \in \mathcal{C}(x)$. The objective of this subproblem is to select the attribute whose associated period 2 choice, i.e. $\arg \min_{x \in \mathcal{C}_i \cap A} u(x)$, has normative rank as close to x as possible. Notice that x may not be choosable in this subproblem (if, for example, each set $\mathcal{C}_i$ contains an element y with $u(y) < u(x)$) in which case the DM treats $u(x)$ as a consumption target and optimally selects $\mathcal{C}_i$ to approximately attain $u(x)$. The quantity, $\max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{y \in \mathcal{C}_i \cap A} u(y)$, is the (optimal) value of the subproblem associated to setting a consumption target of $u(x)$. Let $z_x \in A$ be the choice at which this value is attained. Hence, the utility of a menu represents a DM who trades off normative rank of a consumption target, $u(x)$, against the gap $u(x) - u(z_x)$, which represents the temptation cost due to the shortfall between the utility of the target x and that of the second-best choice z_x. Utility of a menu can (vacuously) be written as $U(A) = \max_{x \in A} [u(x) - (u(x) - u(z_x))] = \max_{x \in A} u(z_x)$.

We now consider some examples which show how categories constrain the DM into picking a second-best z_x when the target is x. In both examples the "choice data" to be explained consists of a pair (u, C), where u is commitment (normative) preference (i.e. the ranking on singleton menus) and $C(\cdot)$ is a choice correspondence. For each example we construct a category model representation, which yields a menu preference whose (i) singleton ranking agrees with u and (ii) whose implied second stage choices agree with $C(\cdot)$. Moreover, by the results in section 4, there is a reversible path connecting menu preferences (satisfying our axioms) and pairs

(u, C) satisfying the reformulated versions of these axioms on the domain of ex post choices. The first example is a temptation-version of the well-known compromise effect, see Simonson and Tversky (1993). The labels on the consumption choices follow Dekel et al. (2009), who use a similar example to motivate their generalization of the Gul and Pesendorfer (2001) model. Fix a triple x, y, z where x is a healthy (slightly sweet) snack, y is a salty snack, and z is a decadently sweet snack. Let $u(\cdot)$ denote the normative ranking, with $u(x) > u(y) > u(z)$. Consider the following choice behavior:

$$C(\{x, y, z\}) = \{y\}, C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{z\}.$$

We find a category model that rationalizes this choice behavior.

Example 1 (Compromise). Put $\mathcal{C}_1 := \{y\}, \mathcal{C}_2 := \{x, z\}$ and $\mathcal{C} \equiv \{\mathcal{C}_1, \mathcal{C}_2\}$. Let $U(\cdot)$ denote the associated menu utility, i.e. $U(A) := \max_{\mathcal{C}_i} \min_{z \in A \cap \mathcal{C}_i} u(z)$. Notice that:

- $U(\{x, z\}) = u(z), U(\{x, y\}) = u(x)$.

- $U(\{x, y, z\}) = u(y)$.

The story driving the example is one of choice induced by "compromise". The DM would not be baited into choosing y over x in the absence of the more decadent option. He only chooses y to mitigate his urge to select the decadent option. In this sense, the choice of y is clearly an act of self-control. Yet, as pointed out in Dekel et al. (2009) this example does not afford a Gul and Pesendorfer (2001) representation. It also does not admit a Dekel et al. (2009) representation, although some more recent extensions, e.g. Noor and Takeoka (2010), of the Gul and Pesendorfer (2001) model can capture this example. With the given story, the sets $\{\mathcal{C}_1, \mathcal{C}_2\}$ comprising the category have the following interpretation. The first set consists of a grouping of all choice objects (snacks) which are salty, and the second is a grouping of all snacks which are sweet (to any degree). In this example, there is only one salty snack insofar as the DM's preferences are concerned. Since the attribute "salty" maps to a singleton choice, by selecting this attribute the DM can avoid the temptation to pick something more (ex ante) harmful so long as y is available.

An important feature of the compromise example is that the presence of the middle option is welfare-improving as it prevents the choice of the more harmful temptation. Now consider a variation of this example. It is similar to compromise in that (i) the normative preference u is the same and (ii) the introduction of a more harmful temptation (z) induces a choice of a lesser temptation (y). However, in contrast to the previous example the presence of y is also harmful – the DM is better off when any temptation is removed since it is the sum total of these temptations that induces the choice distortion. This can be interpreted as a temptation analogue of the well-known attraction effect (a.k.a "asymmetric dominance" effect),

see Simonson and Tversky (1993) for an experiment in which this effect has been documented. One story motivating the attraction effect is salience. The presence of the dominated option makes the dominating option more salient, which accounts for the choice switch. In our example, the story would be that the presence of a worse and unchosen temptation makes the lesser temptation seem more appealing (as it is still tempting and doesn't seem as harmful in comparison), hence inducing the switch. This is the sense in which we think of the example as a temptation-analogue of the attraction effect. This is just one explanation for the attraction effect.[7] There are many recent contributions which together provide a diverse set of rationalizations for this effect, see e.g. Natenzon (2010), Ok et al. (2010), de Clippel and Eliaz (2012), Manzini and Mariotti (2012). Again place the menu preference and second stage choices in the context of the Dekel et al. (2009) example. Put $x =$ a healthy snack, $y =$ a salty snack, $z =$ a sweet snack. Assume also, for the sake of this example, that x is both a little salty and a little sweet, but not enough on either account to be viewed as unhealthy:[8]

$$C(\{x, y, z\}) = \{y\}, C(\{x, z\}) = \{x\}, C(\{x, y\}) = \{x\}.$$

We explain these choices with the following category model representation:

Example 2 (Aggregation). Let $\mathcal{C}_1 := \{x, z\}, \mathcal{C}_2 := \{x, y\}, \mathcal{C} \equiv \{\mathcal{C}_1, \mathcal{C}_2\}$. Let $U(\cdot)$ denote the associated menu utility. Notice that:

- $U(\{x, z\}) = u(x) = U(\{x, y\})$.

- $U(\{x, y, z\}) = u(y)$.

We call this "aggregation" since it is a twist of a similar aggregation example in Dekel et al. (2009). The story is that the DM is tempted by salty or sweet snacks when they are on the menu. He can exert self-control by committing to avoid either excessively salty snacks or excessively sweet snacks, just not both at the same time. Grouping choices by whether they are salty or sweet (resp. $\mathcal{C}_2$ or $\mathcal{C}_1$) he can "choose" ex ante which temptation he will avoid. Anticipating that at the point of consumption he might not be able to control his temptation, by choosing a menu it is as if the DM commits to a choice attribute (as opposed to a consumption choice). The two representations provide an explanation for why the introduction of y is

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[7]Moreover, in our model the attraction is asymmetric as it always pulls away from the u-maximal element.

[8]To be consistent, we could have taken x to be a little of both in the preceding example as well – so that we are looking at the same set of objective choices and only preferences are changing between the two examples. Since attributes only "count" if they can change choices the fact that x is not grouped with y in the first example says that the DM finds the objective characteristic "saltiness" in choice x irrelevant in determining whether he will or won't choose it when it is available.

welfare enhancing in the first example and harmful in the second. Adding y adds an attribute in example 1, hence provides a self-control device which did not exist in y's absence. By contrast, adding y in example 2 is harmful since it shares an attribute with x. Adding it to the choice set dilutes the DM's self-control as he could have obtained x via attribute choice in y's absence.

2.2 Axioms

The main tool to elicit attributes from menu preference (and also to define the axioms) is a concept dubbed "subjective feasibility". To motivate the idea, let us assume the menu preference admits a category representation and say that x is choosable in A via attribute choice. This happens exactly when the consumption target $u(x)$ can be attained on the sub-collection $\mathcal{C}(x)$, so that there is some $\mathcal{C}_i \in \mathcal{C}(x)$ with $x \in \arg\min_{y \in \mathcal{C}_i \cap A} u(y)$. A revealed preference implication of this is that the DM can assure himself a consumption utility of at least $u(x)$ when the option set is A, i.e. $A \succeq \{x\}$. Consequently, $u(x)$ is also a lower bound on consumption utility for any subset of options (from A) which contains x.

Definition 1. An element $x \in A$ is *subjectively feasible* if whenever $A' \subseteq A$ and $x \in A'$, then $A' \succeq \{x\}$.

In words, if the DM can always do at least as well as $u(x)$ whenever x is available, then we infer that x is choosable (i.e. $u(x)$ is attainable on the sub-problem $\mathcal{C}(x)$ in a putative category representation). We use this definition to state our first axiom entitled "CRW", i.e. choice reveals welfare. The existential quantifier in the axiom goes away once we express the axioms in terms of ex post choices (see section 4).

Axiom 1: (CRW) Every menu A possesses an $x \in A$ with the property that:

- $\{x\} \sim A$.

- If $A' \subseteq A$ with $x \in A'$, then $A' \succeq A$.

These two conditions are supposed to reveal that x is chosen from the menu in the second stage. The first and second condition say that every menu possesses a welfare-equivalent which is subjectively feasible. Thinking of x as the choice from the menu, the second condition also says that if x is still available in A' then the DM cannot be worse off since he has the option of choosing x from A'. We require one more axiom for our representation result. Let A^* denote the subset of subjectively feasible elements in a menu A. From Definition 1, we have $A^* := \{x \in A : \forall A' \subseteq A \text{ s.t. } x \in A', A' \succeq \{x\}\}$.

Axiom 2: (Strong Reduction) If $A^* \subseteq A' \subseteq A$, then $A' \sim A$.

The axiom combines two postulates. First, the set of subjectively feasible elements A^* determines the indifference class of the menu. Intuitively, if these are indeed the only elements that can be chosen on account of temptation, then if we remove options that were not choosable in the first place this shouldn't change the DM's welfare. The second postulate is that if there is a subset A' sandwiched in between, then everything feasible in A is feasible in the subset A'. In addition, nothing in A' is feasible that was not already feasible in A, i.e. $(A')^* = A^*$. The first part of this second postulate is itself a sort of contraction consistency condition on feasibility and the second part is a restricted expansion consistency condition: if the contracted menu A' has the property that it contains all the feasible elements of A, then anything that is feasible in A', i.e. $x \in (A')^*$, remains feasible in A.[9]

3   Main Results

3.1   Representation Theorem

The main results of the paper are that order, CRW, and strong reduction characterize the category model and are strong enough to identify categories from menu preferences.

Theorem 1. *A preference $\succeq \, \in \mathbf{P}(\mathrm{X})$ satisfies Axiom 1 (CRW) and Axiom 2 (Strong Reduction) if and only if it admits a category representation.*

A basic ingredient in the construction of the representation is the notion of an aggregation set. Call a menu $A(x)$ an *aggregation set* for x if (i) $\{x\} \succ \{y\}, \forall y \in A(x)$, (ii) $\{x\} \succ \{x\} \cup A(x)$, and (iii) $\{x\} \sim \{x\} \cup (A(x) \backslash y), \forall y \in A(x)$. The difficulty in constructing a category model representation is in accommodating the presence of these aggregation sets. When there are no aggregation sets, the recursive construction we sketch below concludes at the first step of the recursion. In this case, we get what we call "no aggregation" categories. These are an interesting subclass of categories, e.g. they nest Strotzian preferences, Set-Betweenness categories (i.e. preferences satisfying CRW and Set-Betweenness), and so on. We relegate discussion of no aggregation categories and related submodels to a supplement.

Turning to the definition, the first two conditions say that x is not subjectively feasible in the menu $A(x) \cup \{x\}$. The last condition says that $A(x) \cup \{x\}$ is a minimal (w.r.t. set inclusion) set at which x is not subjectively feasible. We summarize the fact that $A(x)$ is an aggregation set for x with the notation "$x \rightarrow_t A(x)$". Also say that $x \rightarrow_t y$ if $\{x\} \succ \{y\}$ and $\{y\} \sim \{x, y\}$. To elicit attributes, we imagine that x is

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[9]In proposition 2 (section 4) we decouple some of these restrictions by considering choice filters induced by category models, viz. one of the conditions on the filters is a version of this restricted expansion condition and the other is a contraction consistency condition.

some unobservable bundle of attributes. These attributes are revealed to the analyst as we offer menus to the DM of the form $\{x\} \cup A(x)$, where $A(x)$ is an aggregation set. Each attribute that describes x serves as a self-control device against possibly tempting options on the menu. Moreover, when there are no other options on the menu which share that attribute then by "choosing" this attribute in stage 1 the DM enables the selection of x in period 2. The fact that the DM *cannot* choose x from the menu $\{x\} \cup A(x)$ means that all attributes that describe x are accounted for by elements of $A(x)$, viz. every attribute (relevant for choice) possessed by x is possessed by some element of $A(x)$.

Were we in the setting of example 2 we would stop here since the elements of the aggregation set are each described by a single attribute. The general construction is complicated by two facts. First, elements of aggregation sets can themselves be bundles of attributes – so that the underlying space of attributes is larger than $A(x)$. Second, there may be more than one aggregation set for x and each element of these aggregation sets can itself be an (implicit) bundle of attributes. To take both possibilities into account, we use a recursive procedure. Using the decomposition $\mathcal{C} \equiv \cup_x \mathcal{C}(x)$, we construct a representing $\mathcal{C}$ by constructing $\mathcal{C}(x)$ "one x at a time". To this end, we start from $(\succeq)|_X$-minimal x and induct upwards on $(\succeq)|_X$-rank. We then use a recursive construction called an "x-tree" to recover the space of attributes that describe x. The tree structure comes out of a repeated "unbundling" process. Let $A(x) := \{y_1, \ldots, y_k\}$ be the aggregation set for x, and create a *coarse* attribute space for x via $\{x, y_1\}, \{x, y_2\}, \ldots, \{x, y_k\}$. Since x is not choosable in $\{x\} \cup A(x)$, this reveals that every attribute possessed by x is also possessed by some y_i. However, the y_i's can themselves be (implicit) bundles of attributes so that each set $\{x, y_i\}$ represents choices which share a set of attributes. In this sense, the collection $\{x, y_i\}$ is a coarse attribute space for x. For each y_i which possesses an aggregation set we "unbundle" the attributes implicitly grouped together in the single set $\{x, y_i\}$ to form the n sets, (put $A(y_i) = \{y_i^1, \ldots, y_i^n\}$), $\{x, y_i, y_i^1\}, \{x, y_i, y_i^2\}, \ldots, \{x, y_i, y_i^n\}$. This yields a tree of sets:
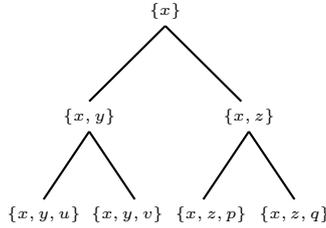

Figure 1: An x-tree with $A(x) = \{y, z\}, A(y) = \{u, v\}, A(z) = \{p, q\}$.

Now iterate the previous step. If any of the y_i^j themselves possess an aggregation set, then the group $\{x, y_i, y_i^j\}$ is also a coarse attribute as it consists of all choices

described by the set of attributes which (implicitly) describe y_i^j. We then use the elements of an aggregation set $A(y_i^j)$ to unbundle the coarse attribute $\{x, y_i, y_i^j\}$. Repeatedly unbundling until we exhaust all aggregation sets, we obtain a collection of sets $\mathcal{C}(x)$. Putting $\{x\}$ as a root node the iterative refinement process splits each (unique) predecessor node into several successor nodes, i.e. the splitting procedure yields a tree of successively refined attributes. Calling this "splitting" might seem a misnomer since at subsequent nodes the sets $\mathcal{C}^i$ are actually getting larger. The splitting refers to the implicit bundle of attributes shared by elements in the set $\mathcal{C}^i$. As these sets get larger, the shared set of attributes shrink – eventually to a singleton, which is when the splitting process stops. The entire set of nodes, starting from the root $\{x\}$ all the way down to the terminal nodes in $\mathcal{C}(x)$ is what we refer to as an "x-tree". Carrying out the x-tree construction for each x we obtain a category $\{\mathcal{T}_x\}_{x \in X}$ which we allege is a representation of the menu preference. The general x-tree construction follows this sketch closely, but it too is complicated by some facts. First, as already noted, choices possess multiple aggregation sets. When we unbundle the coarse attribute $\{x, y_i\}$, we need to account for the fact that there may be several distinct sets $A(y_i)$, each of which represents a bundle of attributes and, hence, each of which is (possibly) culpable for the infeasibility of x in the menu $\{x, y_i\}$.

Second, the axioms imply "transitive infeasibility", i.e. if $x \to_t A(x)$, $z \to_t A(z)$ for $z \in A(x)$, then $\{x\} \succ \{x\} \cup ((A(x) \backslash z) \cup A(z))$. This means that the y-tree is *nested* within the x-tree whenever $\{x\} \succ \{y\}$ and $y \in A(x)$ for some aggregation set $A(x)$. Hence, when we refine, say, $\{x, y_i\}$ we need to account for not just aggregation sets $A(y_i)$, but also aggregation sets $A(z)$ for any $z \in A(y_i)$, and – in turn – aggregation sets $A(w)$ for any $w \in A(z), z \in A(y_i)$, and so on. In other words, when we refine $\{x, y_i\}$ we need to account for the entire y_i-tree of successively refined attribute sets corresponding to each element $y_i \in A(x)$. This forces the construction of categories to be recursive. At the end of the recursion we obtain the collection of sets $\mathcal{T}_x$ for each $x \in X$ which we refer to as the "tree category". The key step in showing representability is in showing that if x is subjectively feasible in A then it is choosable in A via attribute choice. In other words, we need to find a set $\mathcal{C}_i \in \mathcal{T}_x$ with $x \in \arg\min_{y \in \mathcal{C}_i \cap A} u(y)$. Equivalently, we find a path,

$$\ell := \{\mathcal{C}^0 \ (= \{x\}) \ \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^t \ (= \mathcal{C}_i)\}$$

through the x-tree, where for each coarse attribute $\mathcal{C}^j$ on the path x is the only element in A possessing this attribute, i.e. $\{x\} = \arg\min_{y \in \mathcal{C}^j \cap A} u(y)$. Call this an "unobstructed" path.

Construct the first two steps of such a path. Since x is feasible in A, fixing any aggregation set $A(x)$ we know that there is some $z \in A(x)$ with $z \notin A$. That is, if the DM commits to the coarse attribute represented by z (when the option set is A), then

he enables the choice of x. Unbundling the attributes that implicitly describe z, this means that for any aggregation set $A(z)$ there must be some $p \in A(z)$ with $p \notin A$. Note that we are using transitive infeasibility here, viz. if $A(z)$ is an aggregation set, then $\{z\} \succ \{z\} \cup (A(z)\backslash p) \cup A(p)$ – i.e. some subset of $(A(z)\backslash p) \cup A(p)$ is an aggregation set for z. Let $\mathcal{C}^0$ denote the root node $\{x\}$, $\mathcal{C}^1$ denote the coarse attribute $\{x, z\}$, $\mathcal{C}^2$ denote the (coarse) attribute $\{x, z, p\}$. Since $z, p \notin A$ we have an unobstructed path through the first two levels of the x-tree:
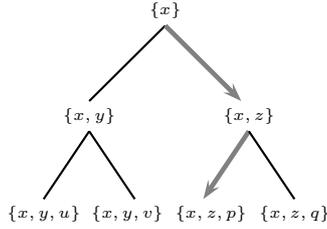

Figure 2: An unobstructed path through part of the x-tree when $A = \{x, y, q\}$.

Successively refine the attribute sets $\{x, z, p\}$ by unbundling p (if necessary) then unbundling elements of aggregation sets of p, and so on. At each unbundling step we find an unobstructed extension of the path ℓ which we concatenate with the existing unobstructed path. Inductively proceeding until aggregation sets are exhausted, we obtain an unobstructed path through the full x-tree, $\ell := \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \cdots \to \mathcal{C}^t$, where $\mathcal{C}^t$ is a terminal node of the x-tree.

3.2   Identification

To identify categories from preferences we will need to place some restrictions on the class of admissible models. First, we exclude ties in the singleton ranking as these provide a trivial source of multiplicity in the set of model representations. For example, say $\{x\} \sim \{x'\}$ and put $x \to_t y, x' \to_t y$. Consider the two category models: $(u, \mathcal{C}(1)), (u, \mathcal{C}(2))$, where $\mathcal{C}(1) \equiv \{\{x, x', y\}\}, \mathcal{C}(2) \equiv \{\{x, y\}, \{x', y\}\}$. Note that both models represent the same menu preference although the attribute sets $\{x, y\}, \{x', y\}$ are only cosmetically distinct. To avoid this issue, we restrict (for the identification result only) to menu preferences in which the singleton ranking is strict. There are two additional ways in which categories could be non-identified due to attribute sets that are in the model $(u, \mathcal{C})$ yet do not "count" towards the representation. The first of these is analogous to the Dekel et al. (2001) notion of a relevant state and is a property that we call "sharpness".

Definition 2. A model $(u, \mathcal{C})$ is *sharp* if for any sub-collection $\mathcal{C}' \subseteq \mathcal{C}$, the induced menu preferences $\succeq_{(u,\mathcal{C})}$ and $\succeq_{(u,\mathcal{C}')}$ are unequal.[10]

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[10]The sub-collection is assumed to satisfy $\cup_i \mathcal{C}'_i = X$, else the statement that $\succeq \neq \succeq'$ is obvious.

The preferences $\succeq_{(u,\mathcal{C})}, \succeq_{(u,\mathcal{C'})}$ denote the menu preferences underlying the model $(u,\mathcal{C})$ (resp. $(u,\mathcal{C'})$). Sharpness requires that all attribute sets are necessary in the following sense: if we delete any one of these sets from the model, then we change the menu preference (underlying the new model). The second way in which attribute sets could be superfluous is a little more subtle. To describe it we first require a definition.

Definition 3. Given two models $(u,\mathcal{C}_1), (u,\mathcal{C}_2)$ say that $(u,\mathcal{C}_2)$ is a *prolongation* of $(u,\mathcal{C}_1)$ if there is a bijection $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ where for every set $\mathcal{C}_1(i) \in \mathcal{C}_1$ we have $\pi(\mathcal{C}_1(i)) \supseteq \mathcal{C}_1(i)$.

When $\mathcal{C}_2$ is a prolongation of $\mathcal{C}_1$ we will equivalently refer to this by saying $\mathcal{C}_1$ is a "retraction" of $\mathcal{C}_2$. We will show that sharp categories are identified up to a specific kind of prolongation. Given a pair of sets (A, B) where $A \subseteq B$ (both contained in X) say that A is a *lower bound order interval* in B if there is some $z \in B$ such that $(-\infty, z] \cap B := \{x \in B : \{z\} \succeq \{x\}\} = A$. Now consider two models $(u,\mathcal{C}_2), (u,\mathcal{C}_1)$ where $\mathcal{C}_2$ prolongs $\mathcal{C}_1$ (under bijection π) and, moreover, $\mathcal{C}_1(i)$ is a lower bound order interval in $\pi(\mathcal{C}_1(i))$ for every $\mathcal{C}_1(i) \in \mathcal{C}_1$. This is the second sense in which categories could be non-identified. While sharpness is a criterion for eliminating redundant attribute sets in the representation, the criterion we have just described is a method for determining redundant *elements* of attribute sets in the representation. To see this, let $x \in \pi(\mathcal{C}_1(i)) \backslash \mathcal{C}_1(i)$ so that x possesses the attribute represented by elements of the set $\pi(\mathcal{C}_1(i))$. This attribute only "counts" in the representation if there is some menu A where x is choosable precisely by committing to this particular attribute, i.e. $x \in \arg\min_{y \in \pi(\mathcal{C}_1(i)) \cap A} u(y)$. However, since $\mathcal{C}_1$ is also a representation of the same menu preference this means there is some $\mathcal{C}_1(j)$ with $x \in \arg\min_{y \in \mathcal{C}_1(j) \cap A} u(y)$, i.e. x is choosable in A via committing to attribute $\mathcal{C}_1(j)$ (using the model $\mathcal{C}_1$).

Since $\mathcal{C}_2$ prolongs $\mathcal{C}_1$ and $\mathcal{C}_1(j)$ is a lower bound order interval in $\pi(\mathcal{C}_1(j)) \in \mathcal{C}_2$ this means that there is more than one way to choose x (in A) via committing to an attribute. Using the model $\mathcal{C}_2$ either the DM can choose attribute $\pi(\mathcal{C}_1(i))$ and enable the second stage choice of x or he can choose attribute $\pi(\mathcal{C}_1(j))$ and induce the same second stage choice. In other words, while x may physically share the attribute represented by the set $\pi(\mathcal{C}_1(i))$ this object description is irrelevant for choice. Hence, after reducing the attribute set $\pi(\mathcal{C}_1(i))$ by deleting x and keeping other attributes unchanged we still have a representation. Putting the sharpness and prolongation conditions together is, therefore, tantamount to requiring that both attributes (i.e. sets $\mathcal{C}_i \in \mathcal{C}$) and object descriptions (i.e. elements of sets $\mathcal{C}_i$) are non-redundant. These are two ways in which one could create multiplicity in the category representations. The question which remains is – are there others? The answer is "no". Once attribute sets and object descriptions are required to be non-redundant (and the singleton ranking is strict), the category model is identified.

Theorem 2. *Assume $(\succeq)|_X$ is strict. There is a unique, sharp model $(u, \mathcal{C}_*)$ such that any other sharp representation, $(u, \mathcal{C})$, is a prolongation of $(u, \mathcal{C}_*)$.[11] Moreover, any prolongation $\pi : \mathcal{C}_* \to \mathcal{C}$ has the property that $\mathcal{C}_i \in \mathcal{C}_*$ is a lower bound order interval in $\pi(\mathcal{C}_i)$.*

We now consider two examples. The first example formally shows that both sharpness and prolongation are necessary for identification. The second illustrates why the identification problem is in general non-trivial. A caution to the reader: the latter example requires some notation which is used in the proof of Theorem 1.

Example 3 (Sharpness $\not\Rightarrow$ Uniqueness). Let $X = \{x \succ y \succ z \succ p \succ q\}$. Assume $A_1(x) = \{z, p\}$, $A_2(x) = \{z, q\}$, and $p \to_t q, y \to_t p$. Put $\mathcal{C}_1 = \{y, p, q\}, \mathcal{C}_2 = \{x, z\}, \mathcal{C}_3 = \{x, p, q\}$ and consider the category model $(u, \mathcal{C})$ given by $\mathcal{C} \equiv \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$. Note that $(u, \mathcal{C})$ is a sharp representation of $\succeq$. Now consider the following prolongation of $\mathcal{C}$. Put $\mathcal{C}'_1 = \{x, y, p, q\}, \mathcal{C}'_2 = \mathcal{C}_2, \mathcal{C}'_3 = \mathcal{C}_3$ and let $\mathcal{C}' \equiv \{\mathcal{C}'_1, \mathcal{C}'_2, \mathcal{C}'_3\}$.

Note that the model $(u, \mathcal{C}')$ also represents $\succeq$ and, importantly, is also sharp. To see this, note that since y is not a member of any aggregation set for x we cannot omit $\mathcal{C}'_3$ from the representation. Also note that the model $(u, \mathcal{C}')$ prolongs $(u, \mathcal{C})$ in the manner described in the theorem: the sets comprising $\mathcal{C}$ are lower bound order intervals of the sets comprising $\mathcal{C}'$. The object description of x using the model $\mathcal{C}'$ (i.e. the sets in the category $\mathcal{C}'$ containing x) is redundant since whenever x is choosable by committing to the attribute $\mathcal{C}'_1$ it is also choosable via committing to $\mathcal{C}'_3$. Thus, we need the prolongation refinement in order to pin down the model.

We conclude with a somewhat complex example. The objective is two-fold. First, to show why the tree category does not, by itself, yield a sharp representation. Second, to give an example of a menu preference complex enough that illustrates the non-triviality of the identification, viz. if there are many attribute sets in a given representation, where some are relevant and others are not, how do we find the unique, minimal set of attributes $\mathcal{C}_*$ whose existence is asserted by the preceding theorem? To this end, we note that the proof of the theorem is constructive. It gives an algorithm that locates the collection $\mathcal{C}_*$.

Example 4. Let $X = \{x, y, z, p, q\}$ where x has aggregation sets $\{y, z\}, \{p, q\}, \{y, q\}$, and $\{p, z\}$. Also put $y \to_t p$ and $z \to_t q$, and assume there are no other constraints on choice. Consider the following category,

$$\mathcal{C}_1 := \{x, y, p\}, \mathcal{C}_2 := \{x, z, q\}$$

where $\mathcal{C}_* \equiv \{\mathcal{C}_1, \mathcal{C}_2\}$. Notice that the pair $(u, \mathcal{C}_*)$ represents $\succeq$. Now compare this with the category we obtain from the algorithm in the theorem, i.e. the construction

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[11]The unique, minimal category $\mathcal{C}_*$ is constructed explicitly in the proof and is a sharp subcategory of the tree category, $(u, \mathcal{T})$, constructed in the proof of Theorem 1.

of the "tree" category $\{\mathcal{T}_x\}_{x\in X}$. It is straightforward to see that the only relevant categories are those in the x-tree. There are 4 levels in the x-tree corresponding to the four x-aggregation sets, $A_1(x) = \{y,z\}, A_2(x) = \{p,q\}, A_3(x) = \{y,q\}, A_4(x) = \{p,z\}$. A list of the categories on each level is as follows: (put $B_t(x) := \{y \in X : x \rightarrow_t y\}$)

1. $\mathcal{L}(0) \equiv \{\mathcal{C}_1^0 = \{x\}\}$ (since $B_t(x) = \emptyset$, the root node of the x-tree contains only the singleton $\{x\}$).

2. $\mathcal{L}(1) \equiv \{\mathcal{C}_1^1 = \{x,y,p\}, \mathcal{C}_2^1 = \{x,z,q\}\}$ (the two nodes are obtained by separately attaching $B_t(y) \cup \{y\} = \{y,p\}$ and then $B_t(z) \cup \{z\} = \{z,q\}$ to the initial node of the tree).

3. $\mathcal{L}(2) \equiv \{\mathcal{C}_1^2 = \{x,y,p\}, \mathcal{C}_2^2 = \{x,y,p,q\}, \mathcal{C}_3^2 = \{x,z,p,q\}, \mathcal{C}_4^2 = \{x,z,q\}\}$ (separately attach p and q to each node in the preceding level – note that $B_t(p) = \emptyset = B_t(q)$).

4. $\mathcal{L}(3) \equiv \{\mathcal{C}_1^3 = \{x,y,p\}, \mathcal{C}_2^3 = \{x,y,p,q\}, \mathcal{C}_3^3 = \{x,y,p,q\}, \mathcal{C}_4^3 = \{x,y,p,q\}, \mathcal{C}_5^3 = \{x,y,z,p,q\}, \mathcal{C}_6^3 = \{x,z,p,q\}, \mathcal{C}_7^3 = \{x,y,p,z,q\}, \mathcal{C}_8^3 = \{x,z,q\}\}$ (separately attach $B_t(y) \cup \{y\}$ and (resp.) $\{q\}$ to each node in the preceding level – note that we include replicas in the tree construction).

5. $\mathcal{L}(4) \equiv \{\mathcal{C}_1^4 = \{x,y,p\}, \mathcal{C}_2^4 = \{x,y,p,z,q\}, \mathcal{C}_3^4 = \{x,y,p,q\}, \mathcal{C}_4^4 = \{x,y,p,z,q\}, \mathcal{C}_5^4 = \{x,y,p,q\}, \mathcal{C}_6^4 = \{x,y,p,z,q\}, \ldots, \mathcal{C}_{15}^4 = \{x,z,p,q\}, \mathcal{C}_{16}^4 = \{x,z,q\}\}$ (the pattern at this stage should be clear, hence we omit the full enumeration of all 16 nodes in level 4 of the tree).

Notice that several of the nodes in the terminal level of the x-tree are identical. However, notice also that the category model we started out with, $(u, \mathcal{C}_*)$, is a sub-category of the terminal nodes of the x-tree. In particular, the sets in $\mathcal{C}_*$ are exactly the minimal (w.r.t set inclusion) nodes in the x-tree. This example points to the identification strategy, i.e. where to look for the model $(u, \mathcal{C}_*)$ whose existence is announced by the theorem. The first step in the argument shows that any category model $(u, \mathcal{C})$ is a prolongation of some sub-category of the tree category, $\{\mathcal{T}_x\}_{x\in X}$. We call this the "embedding" step. The intuition is simple: use the decomposition $\mathcal{C} \equiv \cup_x \mathcal{C}(x)$ and embed each $\mathcal{C}(x)$ in $\mathcal{T}_x$. By representability, each $\mathcal{C}_i \in \mathcal{C}(x)$ must contain some element from each aggregation set $A(x)$. And, similarly, for each of these elements that it contains it must, in turn, contain some member of their aggregation sets, and so on. Hence, at each level of the x-tree we can find a coarse attribute that is contained in $\mathcal{C}_i$ – showing that the category $\mathcal{C}$ prolongs a subcategory of $\{\mathcal{T}_x\}_{x\in X}$. This is the crux of the embedding argument. If the original model is sharp, the embedding procedure finds a sharp submodel of the tree category of which the original model $\mathcal{C}$ is a prolongation.

This would nearly seem to deliver identification, but there is a possible indeterminacy in which sharp submodel of the tree category the model $(u, \mathcal{C})$ prolongs. This is where the main use of sharpness comes in and is also the main challenge in identification, viz. we need to show that there is just one sharp submodel of the tree category that retracts $(u, \mathcal{C})$. There isn't a simple intuition that explains why we get uniqueness here; however, we do at least have a hint as to where to look. As the preceding example suggests, if we look at the submodel consisting of minimal (w.r.t. set inclusion) nodes across all $\mathcal{T}_x$ trees, then we have a category representation. In the example, the set of minimal nodes turns out to be the minimal category $\mathcal{C}_*$. In general, the set of minimal nodes will always provide a representation though it need not be a sharp representation. What turns out to be true, however, is that the minimal representation $\mathcal{C}_*$ lies within the set of minimal nodes in the tree category. Moreover, one can extract from the proof an algorithm that always finds the set $\mathcal{C}_*$ within the sub-category of minimal nodes.

4 Comparison between models

4.1 Comparison with temptation models

The main behavioral postulate in the GP model is their Set-Betweenness axiom:

$$A \succeq B \Rightarrow A \succeq A \cup B \succeq B.$$

GP show that as self-control costs become large the DM's preference for commitment vanishes. We show that a similar connection holds for the category model. This begins with the observation that the category model nests the Strotz model, i.e. we can express the dual-self model with a corresponding "Strotz category". Moreover, just as the Strotz model is characterized by the (degenerate) Set-Betweenness axiom when the DM has no preference for commitment, we find that the Strotz model has an alternative axiomatization via a strengthened version of the CRW axiom. The strengthening of CRW also expresses the fact that the DM never has preference for commitment. Akin to the GP setting, where the model collapses to Strotzian choice when self-control costs get large, in our case the category model collapses to Strotzian choice when the space of self-control variables (i.e. the set of attributes) collapses to a singleton (we will be more clear about what we mean by "singleton attribute" in short order).

Definition 4. An element $x \in A$ is a *Strotzian choice* if it satisfies the following two properties:

- $\{x\} \sim A$.

- If $A' \subseteq A$ and $x \in A'$, then $\{x\} \sim A'$.

The element satisfying these two conditions is the putative choice from the menu. The key feature of the definition is that when the DM is choosing such an x, he never has a preference for commitment. Note that this is exactly where the difference with CRW lies – what is possibly a strict preference in the second condition (under CRW) is always indifference under the definition. In the GP model, this happens precisely when the DM picks the v-maximizer on the menu, i.e. choice is "Strotzian".[12] Let the set of Strotzian choices be denoted as $\Sigma_{\mathrm{ST}}(A)$ and define a restriction that we label *Strong CRW*: If $A \neq \emptyset$, then $\Sigma_{\mathrm{ST}}(A) \neq \emptyset$. By contrast, recall the GP axiom that characterizes the Strotz model, denoted *Degenerate Set-Betweenness* (DSB): For any menus A, B either $A \cup B \sim A$ or $A \cup B \sim B$. When we restrict to menu orders, both axioms are equivalent.

Lemma 1. *A preference $\succeq \in \mathbf{P}(X)$ satisfies DSB if and only if it satisfies Strong CRW.*

When Set-Betweenness degenerates to DSB, all choice is Strotzian in the GP model and there is never preference for commitment. Similarly, when CRW is strengthened to Strong CRW, all choice is Strotzian in the category model and the DM no longer has preference for commitment. Now we find the avatar of Strotz preferences when expressed through a category model.[13] Given a Strotz pair (u, v), we first write down a category $\mathcal{C}_{(u,v)}$ where the model $(u, \mathcal{C}_{(u,v)})$ generates the same utility function on menus as the pair (u, v). That is, $\max_{x \in A_v} u(x) = U(A) = \max_{\mathcal{C}_i \in \mathcal{C}_{(u,v)}} \min_{x \in A \cap \mathcal{C}_i} u(x)$. Fix (u, v) and define the following sets

$$\mathcal{C}_x := \{y \in X : v(y) > v(x), u(x) > u(y)\} \cup \{x\}$$

and let $\mathcal{C}_{(u,v)} := \{\mathcal{C}_x : x \in X\}$.

Observation 1. *Fix a Strotz pair (u, v) and a category model $(u, \mathcal{C}_{(u,v)})$. These two pairs generate the same utility function on menus.*[14]

In the Strotz model, there is just a single temptation whose intensity is represented by the v-function. When coded into a category representation, for each $x \in X$

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[12]The Strotz model is defined by a pair of functions (u, v) which form a utility on menus given by $U(A) := \max_{x \in A_v} u(x), A_v := \arg\max_{x \in A} v(x)$.

[13]It is straightforward to see that Strong CRW implies Strong Reduction, so there must be a category representation.

[14]When introducing the category model, we asserted that ex post Strotzian choice (conditional on attribute choice) and multi-Strotzian choice are all sub-classes of the category model. This follows via the embedding we have just described. Fix a triple $(u, \mathcal{C}, v)$ and say that, on $\mathcal{C}_i \cap A$, choices are determined via $\max_{x \in (\mathcal{C}_i \cap A)_v} u(x)$ (as opposed to $-u$), where $A_v = \arg\max_{x \in A} v(x)$. Replace each $\mathcal{C}_i$ with $\{\mathcal{C}_i^x\}_{x \in \mathcal{C}_i}$ where $\mathcal{C}_i^x := \{y \in \mathcal{C}_i : u(x) > u(y), v(y) > v(x)\}$. Doing this for each $\mathcal{C}_i$ we obtain a category model $(u, \mathcal{C}')$ which represents the same menu preference. A similar argument applies if the ex post choice function were determined via: $\max_{x \in \bigcup_k (\mathcal{C}_i \cap A)_{v_k}} u(x)$, where $(u, \mathcal{C}, \{v_k\}_k)$ is the model. Hence, flexibility in ex post choice can be equivalently captured by flexibility in the space of attributes.

there are several sets $\mathcal{C}_i$ in the category that might contain x but the collection satisfies the following two conditions[15]: (put $\{x\} \succ \{y\}$ and $\mathcal{C}(x) = \{\mathcal{C}_i : \mathcal{C}_i \in \mathcal{C}, x = \sup(\mathcal{C}_i)\}$)

1. $x \in \mathcal{C}_i, y \in \mathcal{C}_j \backslash \mathcal{C}_i \Rightarrow (-\infty, y) \cap \mathcal{C}_i \subseteq \mathcal{C}_j$.

2. $\forall x \in X, \exists \mathcal{C}_x \in \mathcal{C}$ s.t. $(-\infty, x) \cap \mathcal{C}_x \subseteq \mathcal{C}_i, \forall \mathcal{C}_i \in \mathcal{C}(x)$.

We refer to the latter property as a "single attribute" condition. To explain why, recall the decomposition where we expressed the category model as the maximum of the respective maxima from the sub-problems consisting of sets in $\mathcal{C}(x)$, i.e.

$$U(A) = \max_{\mathcal{C}_i} \min_{x \in \mathcal{C}_i \cap A} u(x)$$
$$= \max_{x \in X} \max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{y \in \mathcal{C}_i \cap A} u(y)$$

The interpretation of each sub-problem, $\max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{y \in \mathcal{C}_i \cap A} u(y)$, is that $u(x)$ is the utility target and the DM chooses the attribute in $\mathcal{C}(x)$ that puts him closest to this target. When the category satisfies condition (2) above (which we call a NAG – for "no aggregation" – category in the supplement) this sub-problem is trivial since all sets $\mathcal{C}_i \in \mathcal{C}(x)$ contain a common set $(-\infty, x) \cap \mathcal{C}_x$. In this case the solution to the sub-problem, z_x (recall the notation from section 2), always lies in the set $\mathcal{C}_x \cap A$. Hence, to reach the target $u(x)$ the DM has only one attribute to choose from.

The single attribute condition doesn't by itself imply that the DM is Strotzian. For example, note that it allows compromise effects, e.g. the category constructed in example 1 satisfies this condition. In this case there is a single attribute describing each element (as categories are partitions), however these attributes are distinct so that the representation cannot be recovered from a single numerical measure as in observation 1. To this end, condition (1) says that if $\{x\} \succ \{y\}$ and y does not tempt x, then all consumption choices that tempt x, i.e. $(-\infty, y) \cap \mathcal{C}_x$, and that could potentially tempt y do indeed tempt y. This condition is clearly necessary in order to preclude representations of the compromise effect. Evidently it is also sufficient to imply that the menu preference induced by the category model must be Strotzian, so that there is a v from which we can recover the categories (using the formula in the observation).[16]

We now address the max-min structure in the category model and show that a similar structure is present in all models of temptation which share the "Positive Set-Betweenness" axiom. Recall that this is the moniker given by Dekel et al. (2009)

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[15]In a supplemental appendix, we show that categories that have these two properties characterize Strotzian menu preferences.

[16]The proof of this along with a characterization of categories which satisfy the Set-Betweenness axiom is in the supplement.

to the following half of the Set-Betweenness axiom, $A \succeq B \Rightarrow A \succeq A \cup B$. Fix a cardinal $U : \mathcal{M} \to \mathbf{R}$ and consider the map

$$u_{\min}(x, A) := \min\{U(A') : A' \subseteq A, x \in A'\}.$$

It turns out that the formula, $\widehat{U}(A) = \max_{x \in A} u_{\min}(x, A)$, recovers the utility (as opposed to just the underlying menu preference) on menus if and only if the underlying preference satisfies Positive Set-Betweenness.[17] For the GP model, we interpret the number $u_{\min}(x, A)$ as the value of the most costly self-control problem faced by the DM when he has the option to choose x. The GP utility can be expressed as $U(A) = \max_{x \in A} u_{\min}(x, A)$, but for this expression to have meaning we want to compute the kernel $u_{\min}(x, A)$ explicitly. The kernel $u_{\min}(x, A)$ need not agree with the GP maximand $u(x) - (\max_{z \in A} v(z) - v(x))$. For a menu A, define two subsets $A_1(x), A_2(x)$ to be (resp.) $A_1(x) := \{z \in A : \text{(i) } u(z) + v(z) > u(x) + v(x), \text{(ii) } u(z) \leq u(x)\}, A_2(x) := \{z \in A : \text{(i) } u(z) + v(z) \leq u(x) + v(x), \text{(ii) } u(z) \leq u(x)\}$. Note that $A_1(x)$ is the set of options in A that are (weakly) normatively worse than x yet are chosen over x head-to-head (i.e. these are the "overwhelming" temptations) and $A_2(x)$ consists of those normatively worse elements which lose head-to-head with x.

Observation 2. *Fixing a GP pair (u, v), let $U(\cdot)$ denote the induced utility on menus and $u_{\min}(x, A)$ the associated max-min kernel. We have,*

$$u_{\min}(x, A) = \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}.\text{[18]}$$

The value $u_{\min}(x, A)$ is determined in two steps. First, consider the set of temptations which make it costly, but not overwhelming, to choose x. This is the GP value of the menu $A_2(x)$. Now consider the temptations in A against which it is too costly to choose x. If there aren't any, then the value of $u_{\min}(x, A)$ coincides with $A_2(x)$. If $A_1(x)$ is non-empty then we compare the value of each overwhelming temptation against the most costly choice problem in which x is still chosen, i.e. $A_2(x)$, and pick the minimum from these pairwise minima yielding $u_{\min}(x, A)$.

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[17]Proof: Necessity is straightforward. We check sufficiency. For each $x \in A$ let A_x denote the maximal menu (w.r.t. set inclusion) in the set $\arg\min_{A' \subseteq A, x \in A'} U(A')$. By Positive Set-Betweenness (PSB) the unique such maximal element is the union of all sets in the argmin. We now claim that if A_{x^*} is $\succeq$-maximal in the set $\{A_x : x \in A\}$, then $A_{x^*} \succeq A$. To see this, pick any $x_1 \in A$ and write $A = \cup_{x \neq x_1} A_x \cup A_{x_1}$. By PSB, either $A_{x_1} \succeq A$ or $\cup_{x \neq x_1} A_x \succeq A$. In the former case, we are done. In the latter case, pick any $x_2 \neq x_1$ and consider $A'(:= \cup_{x \neq x_1} A_x) = \cup_{x \neq x_1, x_2} A_x \cup A_{x_2}$. By PSB, either $\cup_{x \neq x_1, x_2} A_x \succeq A' \succeq A$ or $A_{x_2} \succeq A' \succeq A$. In the latter case, we are done. In the former case, iterate the preceding argument. Eventually we arrive at an x with $A_x \succeq A$, which implies $A_{x^*} \succeq A$. OTOH, $A \succeq A_x, \forall x$ by definition of the sets A_x. Hence, $U(A) = \max_{x \in A} u_{\min}(x, A)$.

[18]When we take the maximum of $u_{\min}(x, A)$ we obtain a pair $\{x^*, y^*\}$, where y^* is either the maximizer of v on $A_2(x)$ or the minimizer of u on $A_1(x)$, and $A \sim \{x^*, y^*\}$. This is a well-known fact about the GP model (see lemma 2 in Gul and Pesendorfer (2001)) – that every menu is equivalent to a binary submenu that solves the max-min problem, $U(A) = \max_{x \in A} \min_{y \in A} U(\{x, y\})$.

Now consider a category pair $(u, \mathcal{C})$ and let $U(\cdot)$ be the utility given by the formula $U(A) = \max_{\mathcal{C}_i} \min_{z \in A \cap \mathcal{C}_i} u(z)$. For each $x \in X$, recall that $\mathcal{C}(x) := \{\mathcal{C}_i \in \mathcal{C} : x \in \mathcal{C}_i\}$. That is, $\mathcal{C}(x)$ is the sub-collection of sets in $\mathcal{C}$ that contain x.[19]

Observation 3. *Fix a category model $(u, \mathcal{C})$. The associated max-min kernel is given by:* $u_{\min}(x, A) = \max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{z \in \mathcal{C}_i \cap A} u(z)$.

Putting $\mathcal{C} \equiv \{\mathcal{C}_i\}_i$ we can simply reorganize the sets in $\mathcal{C}$ as $\cup_{x \in X} \mathcal{C}(x)$. It follows that

$$
\begin{aligned}
U(A) &= \max_{\mathcal{C}_i} \min_{z \in \mathcal{C}_i \cap A} u(z) \\
&= \max_{x \in X} \left[\max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{z \in \mathcal{C}_i \cap A} u(z) \right] \\
&= \max_{x \in X} u_{\min}(x, A).
\end{aligned}
$$

This shows that the utility on menus for both the GP model and the category model has a max-min representation, where we interpret the latter as expressing a period 1 pessimism towards the welfare impact of temptation. Moreover, as the following observation shows, this interpretation is not sensitive to the selection of a particular utility kernel.

Observation 4. *Fix any function $u(\cdot, \cdot)$ where $u(x, A) \geq u(x, B)$ whenever $A \subseteq B$. Let $U(A) := \max_{x \in A} u(x, A)$ and put $u_{\min}(x, A) := \min\{U(A') : A' \subseteq A, x \in A'\}$. Then, $u(x, A) \leq u_{\min}(x, A), \forall x \in A, \forall A$.*

In other words, fixing the cardinal utility and interpreting $u(x, A)$ as the value of the sub-problem where x is the consumption target, the number $u_{\min}(x, A)$ function is an upper bound on this value across *all* functions $u(\cdot, \cdot)$ (for every x and for every menu A) which generate the same utility on menus, with the caveat that the kernel $u(\cdot, \cdot)$ is non-increasing in the menu argument (the caveat being both necessary and sufficient for Positive Set-Betweenness).

Let us conclude by looking at two extensions of the GP model that also invoke Positive Set-Betweenness. The first is the Dekel et al. (2009) extension, which is comprised of an $(n+1)$-tuple $(u, \{v_k\}_{k=1}^{n+1})$ assembled into a menu utility as follows:

$$
U(A) = \max_{x \in A} \left[u(x) - \sum_{k=1}^{n} c_k(x, A) \right],
$$

where $c_k(x, A) := \max_{y \in A} v_k(y) - v_k(x)$. This is actually a "no uncertainty" version of the model they axiomatize – one of their theorems shows that the model with

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[19]The reader will note that we have switched the definition of the collection $\mathcal{C}(x)$, viz. in the discussion in section 2 we defined $\mathcal{C}(x)$ to be those attribute sets within which x was the top-ranked singleton. On account of identification some (or all) of these sets may be redundant if they are nested within y-trees for some $\{y\} \succ \{x\}$. For this reason, we require the more general definition of $\mathcal{C}(x)$.

uncertainty reduces to the displayed one if and only if Positive Set-Betweenness is satisfied. Another extension of Gul and Pesendorfer (2001) is the Noor and Takeoka (2010) model of non-linear costly self-control,

$$U(A) = \max_{x \in A} [u(x) - \psi(\max_{y \in A} v(y)) \cdot (\max_{y \in A} v(y) - v(x))],$$

where $\psi(\cdot)$ is an increasing non-negative (distortion) function. As mentioned earlier, our examples 1 and 2 are twists of related examples in Dekel et al. (2009) which they used to motivate their generalization of Gul and Pesendorfer (2001). Dekel et al. (2009) also note that the compromise effect (example 1) eludes representation by their model.

Noor and Takeoka (2010) point out that to capture this effect it is necessary for ex post choice to violate WARP – which is captured in their model by allowing violations of independence (resulting in the distortion term $\psi(\cdot)$). Neither the Dekel et al. (2009) nor Noor and Takeoka (2010) extension can represent aggregation effects (unless the menu has three elements). To see this, we consider ex post choices and show that these cannot be captured by the implied second stage choices in these models. Fix $u(x) > u(y_1) \geq u(y_2) \geq \cdots \geq u(y_k)$, where the y_i are temptations. Temptation distorts choices via aggregation when we have:

$$C(\{x\} \cup \{y_i\}_{i=1}^k) = \{y_1\} \ \& \ C(\{x\} \cup A) = \{x\}, \forall A \subsetneq \{y_1, \ldots, y_k\}.$$

The Dekel et al. (2009) model cannot capture these choices since second stage choice is generated by maximization of $u + \sum_{i=1}^n v_i$, and is hence rational. For $k \geq 3$, the Noor and Takeoka (2010) model cannot capture this either since it shares with GP the property that every menu is indifferent to the binary subset consisting of the choice from the menu and its v-maximal element (which must be distinct under either compromise or aggregation). This implies that choice from any subset is the same so long as it contains the original choice and the v-maximal element – a requirement that is violated by aggregation.

4.2 Comparison with ex post choice models

While choices from menus are implied by the category model, we don't directly observe them. However, since temptation distorts welfare only through choices (in our model) it is natural to ask the following converse question: Were we to assume instead that we observe choices from menus (i.e. second stage choice), under what circumstances can we say that these choices were generated by a category model?[20] It is important to be able to map ex post choice data back into the class of menu choice models that we presume to explain these choices. Otherwise, layering a menu

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[20]I thank Wolfgang Pesendorfer and an anonymous referee for suggesting this question.

preference on top of the choice data we want to explain might make the category model seem *ad hoc*. By recovering the same model from choices from menus, we show that the assignment of menu preferences is determined.[21]

Our analysis in this section uses an extra observable. Most models of ex post choice only require an ex post choice correspondence, but we will assume a pair (u, C) consisting of a ranking on consumption choices as well as a choice correspondence.[22] With a temptation story in mind, $u(\cdot)$ is a representation of the DM's commitment preference. This might be overtly observable in cases where it represents a shared norm, e.g. all DM's view smoking cigarettes and excessive risk taking as harmful habits. If there is any temporal lag between committing to consume and consuming, it can also be elicited by offering the DM choices between consumption commitments, i.e. singleton menus. However u happens to be revealed, the objective is to study DM's who, while sharing the same u, respond differently to temptation and, hence, make different choices. By expressing axioms in terms of ex post choices we obtain an equivalent description of the model in terms of "choice filters". This allows us to compare choice behavior derived from the category model with models from the recent literature on quasi-rational (equiv. boundedly rational) choice. For example, the models axiomatized in (resp.) Manzini and Mariotti (2007), Lleras et al. (2008), Manzini and Mariotti (2012), Masatlioglu et al. (2012), Cherepanov et al. (2013) fall under the rubric of the following general model:

$$U(A) = \max_{x \in \Theta(A)} u(x)$$

The correspondence $\Theta : A \rightrightarrows A$ (where $\Theta(A) \subseteq A$) is what is referred to as a choice filter. In Lleras et al. (2008), Masatlioglu et al. (2012), and Cherepanov et al. (2013), the models are expressed in the language of choice filters, with the main differences arising from the conditions imposed on these filters.

For Manzini and Mariotti (2007), Manzini and Mariotti (2012), the filters are a reduced form expression of their original models. The rational shortlist method model (RSM) proposed in Manzini and Mariotti (2007) consists of two binary relations (P_1, P_2). Fixing a menu A, choice maximizes P_2 against the residual set of P_1 maximizers on A. Hence, under RSM, the filter on A would be the residual set of P_1 maximizers on A. In Manzini and Mariotti (2012) (CTC), the model consists of two relations $(\succeq, P_2)$ where $\succeq$ is a binary relation on menus and P_2 is a binary relation on choices. Under CTC, fixing a menu A, in the first stage the DM uses $\succeq$ to only pick elements $x \in A$ which lie in $\succeq$-undominated menus (i.e. undominated categories borrowing the language from Manzini and Mariotti (2012)). Choice is then

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[21]More precisely, what is proven is that the model is recoverable from a pair (u, C) consisting of a commitment ranking on singleton menus and a period 2 choice correspondence.

[22]For brevity we confuse the ordinal ranking with a cardinal representation u.

obtained by maximizing P_2 on the residual elements of A which are drawn from $\succeq$-undominated menus, so that the filter in this case is the set of elements which lie in $\succeq$-undominated menus (contained in A). The category model also induces a choice filter via the formula,

$$\Theta(A) = \cup_i \arg\min_{x \in \mathcal{C}_i \cap A} u(x).$$

The implied ex post choices would be, $C_{(u,\mathcal{C})}(A) = \arg\max_{x \in \Theta(A)} u(x)$. Using the description of the category model in terms of the observables (u, C), we find the properties on choice filters which determine when they derive from category models $(u, \mathcal{C})$.[23]

Definition 5. An element $x \in A$ is *subjectively feasible* if it satisfies the following property. For any $A' \subseteq A$ with $x \in A'$, $u(z) \geq u(x), \forall z \in C(A')$.

The interpretation of the condition is identical to the one given in Definition 1. For a given menu A, let A^* denote the subset of subjectively feasible elements. Recast the axioms as follows.

A1: $u(x) = u(y), \forall x, y \in C(A)$.

A2: If $A^* \subseteq A' \subseteq A$, then $C(A') = C(A)$.

A3: If $A' \subseteq A$ and $C(A) \cap A' \neq \emptyset$, then $u(x) \geq u(y), \forall x \in C(A'), \forall y \in C(A)$.

A1 and A3 are together the analogue of CRW and A2 is the analogue of Strong Reduction. The intuition for A1 is that if choices maximize welfare, then it must be that all elements of $C(A)$ yield the same welfare value. A2 is nearly identical to Strong Reduction. The intuition for A3 is that, if choice is constrained by temptation, then when we pass to a submenu but keep at least one of the original choices available, welfare is weakly improved since this choice should still be feasible in the submenu. Let $\Sigma_{\mathbf{P(X)}}$ denote the subset of menu orders which satisfy Axioms 1 and 2. Let $\Sigma(X)$ denote the set of pairs (u, C) consisting of (i) a utility u on consumption choices and (ii) a choice correspondence C, where the pair (u, C) satisfies A1-A3. Define a map $\Lambda : \Sigma_{\mathbf{P(X)}} \to \Sigma(X)$ as follows. Given $\succeq$, let u be a representation of the order on singleton menus. Second, let $C(A) = \arg\max_{x \in A^*} u(x)$. Put $\Lambda(\succeq) := (u, C)$.

Proposition 1 (Equivalence of Observables). *The sets $\Sigma(X)$ and $\Sigma_{\mathbf{P(X)}}$ are bijectively equivalent under the map Λ.*

It is possible for two distinct menu preferences with a common singleton ranking to induce the same set of ex post choices. The content of the proposition is that

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[23]I thank Faruk Gul for suggesting this result.

for menu preferences which satisfy Axioms 1-2 this cannot happen. We use this fact to find an intrinsic characterization of choice filters $\Theta(\cdot)$ that are induced by category models. Denote a filter induced by a category $(u, \mathcal{C})$ by $\Theta_{(u,\mathcal{C})}(\cdot)$, so that we have $\Theta_{(u,\mathcal{C})}(A) = \cup_i \arg\min_{x \in \mathcal{C}_i \cap A} u(x)$. When this equality holds we say that the choice filter *comes from a category model*. Consider the following restrictions on an abstract filter $\Theta(\cdot)$. The first two are direct analogues of expansion/contraction consistency conditions from choice theory and the third condition is exactly the part of Strong Reduction that pertains to expansion consistency (see our discussion of Strong Reduction when we introduced this axiom).

1. (Sen's α) If $A \subseteq B$, then $\Theta(B) \cap A \subseteq \Theta(A)$.

2. (Restricted Sen's β, I) If $A \subseteq B$ and $\min_{x \in B \setminus A} u(x) \geq \max_{x \in A} u(x)$, then whenever $x, y \in \Theta(A)$ and $y \in \Theta(B)$ we have $x \in \Theta(B)$ as well.

3. (Restricted Sen's β, II) If $A \subseteq B$ and $\Theta(B) \subseteq A$, then whenever $x, y \in \Theta(A)$ and $y \in \Theta(B)$ we have $x \in \Theta(B)$ as well.

4. (Worst choices are always feasible, WCF) If $z \in A$ and $u(y) \geq u(z), \forall y \in A$, then $z \in \Theta(A)$.

The first condition says that if a choice x is feasible in a menu B, i.e. it is an element of $\Theta(B)$, then it is still feasible in a subset A. Since we may have removed some temptations in passing to the subset A but could not have added any new temptations, what is temptation-constrained feasible in B remains so in A. The second condition says that whenever we add options that normatively dominate everything in A, then objects feasible in A remain feasible in B (since we usually only think of y as a temptation for x if $u(x) > u(y)$). The third condition says that if we reach B from A by just adding infeasible elements, then anything feasible in A remains so in B. Again, the intuition is based on temptation: If we add an infeasible element, the cause for its infeasibility is a temptation (or menu of temptations) which was already present in A. In this sense, the restriction is saying that this infeasible element cannot constitute a "new" temptation which was not already present in A.[24] Hence, if $x \in A$ is feasible in the presence of the temptations in A, it remains feasible in B.

Proposition 2. *A filter Θ comes from a category model $(u, \mathcal{C})$ if and only if, fixing this u, it satisfies Sen's α, the restricted Sen's β conditions, and WCF.*

The Lleras et al. (2008) model imposes Sen's α on the choice filter, hence filters which arise from category models are a sub-class of this class of filters. Filters which come from categories are also nested in the Manzini and Mariotti (2012),

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[24]This can be interpreted as a version of the "transitivity of infeasibility" property that we mentioned in sketching the elicitation of categories.

Masatlioglu et al. (2012), and Cherepanov et al. (2013) models. In the first of these, the nesting can be seen from functional forms. For the latter two papers, this follows from the behavioral characterization – as second stage choices from the category model satisfy the relaxation of WARP which is the main axiom that characterizes these models. Turning to the RSM model (Manzini and Mariotti (2007)), there are some sub-classes of the category model that admit an RSM representation, e.g. for special cases such as categories induced by Strotz preferences or categories which are partitions there is an equivalent expression in terms of an RSM model – as pointed out in Horan (2011). Once we allow aggregation effects there is no RSM representation.[25]

5 Conclusion

This paper develops a new axiomatic model of menu choice induced by temptation. The goal is to model behavior where background (and, in some cases, unchosen) temptations nudge the decision-maker into choosing milder temptations which he would not otherwise have chosen. We explain this behavior with a model of (implicit) attribute choice. The idea is that choice objects are bundles of attributes, and this allows the DM to use attributes as a commitment device to avoid certain temptations, i.e. by "choosing" a certain attribute he avoids temptations which don't possess that attribute. Since attributes are subjective, the exercise of the paper is to elicit and identify attributes from menu choice data. Moreover, the model turns out to be recoverable from "ex post" observables, viz. observing (i) the ranking on singleton menus (i.e. normative preference) and (ii) choices from menus is sufficient to elicit and identify the attribute-based model (on menus) which generated these choices.

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[25]Consider the implied choices from the temptation-driven attraction effect (Example 2) and towards contradiction allege an RSM-style representation (P_1, P_2), where P_1, P_2 is a pair of asymmetric binary relations. Note that $C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{x\}$ implies that x survives the elimination round using P_1 since when the choice set is either $\{x, y\}$ or $\{x, z\}$, the choice is x. Hence, $\neg(zP_1x) \wedge \neg(yP_1x)$. OTOH, since $C(\{x, y, z\}) = \{y\}$, this means that x is eliminated in the second stage (the choice round), so that $(yP_2x) \vee (zP_2x)$. Either case contradicts the fact that x is chosen from $\{x, y\}$ (resp. $\{x, z\}$).

6 Appendix

6.1 Proofs for Section 3

First some preliminaries. Introduce an auxiliary function,

$$u_{\min}(x, A) := \min\{U(A') : A' \subseteq A, x \in A'\}.$$

Here we take $U(\cdot)$ to be any cardinal representation of the menu order $\succeq$. Note that the function $u_{\min}(\cdot, \cdot)$ yields a family of rankings $\{\succeq_A\}_{A \in \mathcal{M}}$ that is independent of the choice of $U(\cdot)$. Recall the following axiom, which is "one-half" of the Set-Betweenness axiom introduced in Gul and Pesendorfer (2001): $A \succeq B \Rightarrow A \succeq A \cup B$. Note that this property, referred to as "Positive Set-Betweenness" (PSB) in the literature, is implied by CRW. Introduce two pieces of notation which will be useful in the proof of Theorems 1 and 2. First, let A_x denote the maximal menu (w.r.t. set inclusion) such that $U(A_x) = u_{\min}(x, A)$ (existence follows from PSB). Second, put $\theta_x(A) := \{y : x \succeq_A y\}$. A key property of these "local" menu preferences $\succeq_A$ is that they satisfy the following condition:

$$(*) \ A_x = \theta_x(A).$$

To prove this, note that if $x \succeq_A y$, then $A_x \succeq A_y$. This implies, by PSB, that $A_x \succeq A_x \cup A_y$. Maximality then implies $A_x = A_x \cup A_y$. Thus, $\theta_x(A) \subseteq A_x$. To check the reverse containment, let $y \in A_x$ with $y \succeq_A x$. Then, since $y \in A_x$, we obtain $u_{\min}(y, A) = U(A_y) \le U(A_x) = u_{\min}(x, A)$. On the other hand, $y \succeq_A x$ implies, by definition, $u_{\min}(y, A) \ge u_{\min}(x, A)$ so that $y \sim_A x$.

Let $I_A^1, \ldots, I_A^k$ denote a top-down enumeration of the $\succeq_A$-indifference classes and let $\Sigma(A)$ denote the set of implied choices in A, i.e.

$$\Sigma(A) := \{x \in A : (i) \ \{x\} \sim A, (ii) \ x \in A' \subseteq A \Rightarrow A' \succeq A\}.$$

We check that $\Sigma(A) = \inf\{y : y \in I_A^1\}$. This connects the set of choices with the revealed relation $\succeq_A$.

Lemma 2. $\Sigma(A) = \inf\{y : y \in I_A^1\}$.

Proof. First observe that $\Sigma(A) \subseteq I_A^1$: Otherwise, if $x \in \Sigma(A) \cap I_A^j$ for some $j > 1$, then by CRW we obtain $\cup_{j \ge 2} I_A^j \succeq A$. On the other hand, by $(*)$, $A = \cup_{j=1}^k I_A^j \succ \cup_{j \ge 2} I_A^j$ – contradiction. Thus, $\Sigma(A) \subseteq I_A^1$. Now for any $y \in I_A^1$ consider the menu $A' := \{y, I_A^2, \ldots, I_A^k\}$ and note that $A' \succeq A$, again by definition of $\succeq_A$ and $(*)$. Take any $x \in \Sigma(A)$ so that we obtain: $A' \succeq A \sim \{x\}$. On the other hand, we claim that $u_{\min}(y, A') = U(A')$. To see this, observe that $u_{\min}(x, A') = u_{\min}(x, A), \forall x \in A' \backslash y$ by $(*)$. Since $u_{\min}(y, A) > u_{\min}(x, A), \forall x \in A' \backslash y$ it follows that

$$u_{\min}(y, A') \ge u_{\min}(y, A) > u_{\min}(x, A) = u_{\min}(x, A'), \text{ for all } x \in A' \backslash y.$$

Therefore, by $(*)$, we must have $u_{\min}(y, A') = U(A')$. Thus, we obtain

$$U(\{y\}) = u_{\min}(y, \{y\}) \geq u_{\min}(y, A') = U(A') \geq U(A) = U(\{x\}).$$

It follows that $\{y\} \succeq \{x\}, \forall y \in I_A^1$, implying that $\Sigma(A) \subseteq \inf\{y : y \in I_A^1\}$. To show the reverse containment, simply note that if $x \in \inf\{y : y \in I_A^1\}$ and $x \in A' \subseteq A$, then by definition of $\succeq_A$ and $(*)$ we have $A' \succeq A$. $\qquad\square$

Proof of Theorem 1. Necessity of Axiom 1 (CRW) is straightforward, hence we omit the argument. For necessity of Axiom 2 (Strong Reduction), let $(u, \mathcal{C})$ be a category model and let $\succeq$ denote the underlying menu preference represented by $(u, \mathcal{C})$. Let $A^* = \{x \in A : A' \succeq \{x\}, \forall A' \subseteq A, x \in A'\}$ be the subjectively feasible subset of A, and let $A^*_{(u,\mathcal{C})} = \cup_i \arg\min_{x \in \mathcal{C}_i \cap A} u(x)$, where we take $\mathcal{C} \equiv \{\mathcal{C}_i\}$. Note that Axiom 2 is implied by the equality

$$A^* = A^*_{(u,\mathcal{C})}$$

To check the equality, consider the left-to-right inclusion. Let $x \in A^*$ and put $\Sigma(x) = \{\mathcal{C}_i \in \mathcal{C} : x \in \mathcal{C}_i\}$. We claim that there must be some $\mathcal{C}_i \in \Sigma(x)$ such that $x \in \inf(\mathcal{C}_i \cap A)$. Else, for each $\mathcal{C}_i \in \Sigma(x)$ choose some $z_i \in \inf(\mathcal{C}_i \cap A)$ with $\{x\} \succ \{z_i\}$ and consider the menu $A' := \{z_i : z_i \in \mathcal{C}_i\} \cup \{x\}$. Note that $\{x\} \succ A'$ – contradicting the fact that $x \in A^*$. Thus, there is some $\mathcal{C}_i \in \mathcal{C}$ such that $x \in \inf(\mathcal{C}_i \cap A) \subseteq A^*_{(u,\mathcal{C})}$. For the right-to-left inclusion take $x \in A^*_{(u,\mathcal{C})}$ and take any set $A' \subseteq A$ with $x \in A'$. Find $\mathcal{C}_i \in \mathcal{C}$ such that $x \in \inf(\mathcal{C}_i \cap A)$ and note that this implies $x \in \inf(\mathcal{C}_i \cap A')$. It follows that $A' \succeq A$, implying that $x \in A^*$. Now take $A^* \subseteq A' \subseteq A$ and substitute $A^* = A^*_{(u,\mathcal{C})}$. Notice that this implies $(A')^*_{(u,\mathcal{C})} = A^*_{(u,\mathcal{C})}$. Hence, $A' \sim (A')^*_{(u,\mathcal{C})} = A^*_{(u,\mathcal{C})} \sim A$.

We now turn our attention to the sufficiency of the axioms. To each $x \in X$ we associate an "x-tree". The terminal nodes in each x-tree will be the elements of the overall category.

Step 1: Constructing x-trees.
Introduce the following terminology. Fix an index set $\{1, 2, \ldots, N\}$. An x-tree is a triplet of data $(\{\mathcal{C}_j^i\}_{j=1}^{n_i}, \{\mathcal{L}_i\}_{i=1}^N, \{\mathcal{C}_j^i \to \mathcal{C}_k^{i+1}\})$, consisting (resp.) of nodes, levels, and branches, with the following structure:

- A collection of *nodes* $\mathcal{C}_1^i, \mathcal{C}_2^i, \ldots, \mathcal{C}_{n_i}^i$ for each index i.

- A collection of *levels* $\mathcal{L}(1), \ldots, \mathcal{L}(N)$, where each $\mathcal{L}_i := \{\mathcal{C}_1^i, \mathcal{C}_2^i, \ldots, \mathcal{C}_{n_i}^i\}$

- A collection of *branches* $\{\mathcal{C}_j^i \to \mathcal{C}_k^{i+1}\}$ connecting nodes on consecutive levels. Call $\mathcal{C}_j^i$ the *root* of the branch $\{\mathcal{C}_j^i \to \mathcal{C}_k^{i+1}\}$.

- Every node $\mathcal{C}_k^i$ in level i (for $i > 0$) has a unique root in level $i - 1$.

- Every node $\mathcal{C}_k^i$ in level i is the root of a branch.

Using these objects, we inductively construct an x-tree as follows. First, make the following simplification. Since representability requires that $B_t(x) \in \mathcal{C}_i$ whenever $x \in \mathcal{C}_i$, we make no distinction between the element $\{x\}$ and the set $\{x\} \cup B_t(x)$. That is, whenever we say $x \in \mathcal{C}_i$ what we implicitly mean (unless explicitly stated otherwise) is that $B_t(x) \cup \{x\} \subseteq \mathcal{C}_i$. Let $A_1(x), A_2(x), \ldots, A_n(x)$ enumerate the aggregation sets of x (here we distinguish between $\{x\}$ and $B_t(x) \cup \{x\}$). The x-tree construction proceeds by double-induction. The outer induction is on the $\succeq$-rank of the element x for which the x-tree has been constructed. The inner induction is on the tree construction for a fixed element x. For a $(\succeq)|_X$-minimal element x (in X), let the x-tree be just the singleton node $\{x\}$. Taking this as the base step of the outer induction, induct upwards on $(\succeq)|_X$-rank to construct an x-tree as follows. Let $A_i(x) = \{x_1^i, \ldots, x_k^i\}$ be the elements of the aggregation set and take

$$\mathcal{C}_i^1 = \{x, x_1^i\}, \mathcal{L}_1 = \{\mathcal{C}_1^1, \mathcal{C}_2^1, \ldots, \mathcal{C}_k^1\}$$

Note that the sizes of the aggregation sets $A_i(x)$ need not be the same. For notational brevity, we suppress this dependence – it will make no difference whatsoever for the ensuing arguments. Inductively, assume we have defined nodes and branches (with unique root restriction) for levels $\{1, 2, \ldots, m\}$ (for $m \leq n$ – where n is the total number of aggregetion sets for x) and define level $m + 1$ as follows. Let $\{\mathcal{C}_i^m\}_{i=1}^{N_m}$ be an enumeration of the nodes that form $\mathcal{L}_m$. For each node $\mathcal{C}_i^m$ create $|A_{m+1}(x)|$ branches as follows. Let $A_{m+1}(x) = \{x_1^{m+1}, x_2^{m+1}, \ldots, x_k^{m+1}\}$ and put $\mathcal{C}_1^{m+1} = \mathcal{C}_1^m \cup \{x_1^{m+1}\}, \mathcal{C}_2^{m+1} = \mathcal{C}_1^m \cup \{x_2^{m+1}\}, \ldots, \mathcal{C}_k^{m+1} = \mathcal{C}_1^m \cup \{x_k^{m+1}\}$. Similarly, put $\mathcal{C}_{(i-1)k+1}^{m+1} = \mathcal{C}_i^m \cup \{x_1^{m+1}\}, \mathcal{C}_{(i-1)k+2}^{m+1} = \mathcal{C}_i^m \cup \{x_2^{m+1}\}, \ldots, \mathcal{C}_{ik}^{m+1} = \mathcal{C}_i^m \cup \{x_k^{m+1}\}$. Thus, level $\mathcal{L}(m+1)$ consists of $N_m \cdot |A_{m+1}(x)|$ nodes, $\mathcal{C}_j^{m+1}$, and $N_m \cdot |A_{m+1}(x)|$ branches, $\{\mathcal{C}_i^m \to \mathcal{C}_j^{m+1}\}$ (where $(i-1) \cdot k + 1 \leq j \leq i \cdot k$). Inductively proceed until we exhaust all of the aggregation sets $\{A_1(x), \ldots, A_n(x)\}$. Let $\mathcal{L}(n) = \{\mathcal{C}(1), \mathcal{C}(2), \ldots, \mathcal{C}(N)\}$ be an enumeration of the nodes at level $\mathcal{L}(n)$. For the next step, find the $(\succeq)|_X$-maximal y such that $\{x\} \succ \{y\}$ and for each $\mathcal{C}(i)$ with $y \in \mathcal{C}(i)$ attach a y-tree (which has been constructed by the induction hypothesis). This extends the levels in the original x-tree by the number of levels in the y-tree. For each $\mathcal{C}(i)$ in level $\mathcal{L}(n)$ that does not contain y we just extend a single branch $\{\mathcal{C}(i) \to \mathcal{C}_{n+1}(j_i)\}, \{\mathcal{C}_{n+1}(j_i) \to \mathcal{C}_{n+2}(k_i)\}$, and so on, for each subsequent level, where we put $\mathcal{C}(i) = \mathcal{C}_{n+1}(j_i) = \cdots = \mathcal{C}_{n+M}(l_i)$ (here we take M to be the number of levels in a y-tree). Thus, we obtain a tree with $n + M$ levels. Now continue this procedure. Take a $(\succeq)|_X$-maximal z with $\{y\} \succ \{z\}$ and for each $\mathcal{C}(i) \in \mathcal{L}(n + M)$ with $z \in \mathcal{C}(i)$ attach a z-tree. Iteratively proceed as above. Since X is finite, this process terminates at some level $\mathcal{L}(N_x)$. This concludes the construction of the x-tree.[26]

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[26] If there are no ties in the singleton ranking, this construction is canonical. If there are ties, then we fix a top down labeling $\{x_1, \ldots, x_n\}$ at the outset and at each step of the recursion pick a $(\succeq)|_X$-maximal z. When there are ties, break the tie according to the pre-selected labeling.

Step 2: Check the inequality $U^{\mathcal{C}}(\cdot) \leq U(\cdot)$.

Having defined the x-tree for each $x \in X$ we take the category, $\mathcal{C}$, to be the set of all terminal nodes in the level $\mathcal{L}(N_x)$ for every $x \in X$. Taking u to be a representation of the singleton ranking, the claim is that the pair $(u, \mathcal{C})$ represents $\succeq$. Let $U(\cdot)$ be any cardinal representation of $\succeq$ (which extends u) and let $U^{\mathcal{C}}(\cdot)$ denote the utility defined by the category formula for the pair $(u, \mathcal{C})$, where $u \equiv U(\cdot)|_X$. We show representability by checking equality $U^{\mathcal{C}}(\cdot) = U(\cdot)$ on all menus. The tree structure of the categories allows for a useful decomposition of the function $U^{\mathcal{C}}(\cdot)$. Let $\{\mathcal{L}(N_y)\}$ denote the set of all terminal levels across all trees. Abusing notation, let $\mathcal{T}_{y_1}, \mathcal{T}_{y_2}, \ldots, \mathcal{T}_{y_n}$ be an enumeration of all trees (where $|X| = n$) and define

$$U^{\mathcal{T}_y}(A) := \max_{\mathcal{C}_i \in \mathcal{L}(N_y) : \mathcal{C}_i \cap A \neq \emptyset} \min_{z \in A \cap \mathcal{C}_i} u(z)$$

As before, we suppress the requirement that we maximize only over terminal nodes $\mathcal{C}_i$ which intersect A. Observe that we have the equality

$$U^{\mathcal{C}}(A) = \max_{x \in X} U^{\mathcal{T}_x}(A)$$

Thus, the value the category utility assigns to menu A is the maximum of its value across trees. We analyze $U^{\mathcal{C}}(\cdot)$ by analyzing its behavior on a tree-by-tree basis. In particular, we check that for each x we have $U^{\mathcal{T}_x}(A) \leq u_{\min}(x, A)$. If $u_{\min}(x, A) = u(x)$, then this claim is obvious since we clearly have $\min_{z \in \mathcal{C}_i \cap A} u(z) \leq u(x)$ for each terminal node $\mathcal{C}_i$ of the $\mathcal{T}_x$-tree. Consider the case where $u_{\min}(x, A) < u(x)$. For a given tree $\mathcal{T}_x$, let $\mathcal{L}(1), \ldots, \mathcal{L}(N_x)$ denote its levels. For each level $\mathcal{L}(i)$ consider the function

$$U^{\mathcal{L}(i)}(A) := \max_{\mathcal{C}_j \in \mathcal{L}(i)} \min_{z \in A \cap \mathcal{C}_j} u(z).$$

Note that the menu A contains either a singleton temptation or a non-trivial aggregation set $A(x)$ for x. Consider the latter case and note that by Strong Reduction and the minimality property of $A(x)$, we may find $z \in \sup(A(x))$ such that $\{x\} \cup A(x) \sim \{z\}$. Let $\mathcal{L}(i)$ denote the level at which this aggregation set is introduced into each node of the $\mathcal{T}_x$-tree. We then have

$$(*) \ U^{\mathcal{L}(i)}(A) \geq U^{\mathcal{L}(i+1)}(A) \geq \cdots \geq U^{\mathcal{L}(N_x)}(A) = U^{\mathcal{T}_x}(A)$$

Note that $U^{\mathcal{L}(i)}(A) \leq u(z)$, so that $U^{\mathcal{T}_x}(A) \leq u(z)$. This holds for all aggregation sets $A(x) \subseteq A$.

Now we construct a particular aggregation set $A_*(x)$ with $U(\{x\} \cup A_*(x)) = u_{\min}(x, A)$. Consider the orders $\succeq_A$ underlying the function $u_{\min}(\cdot, A)$.[27] Let $I_A(x)$

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[27]This part of the argument makes essential use of structural properties of these orders $\succeq_A$ under axiom CRW, viz. lemma 2

31

denote the $\succeq_A$-indifference class of x and let $I_A^1, \ldots, I_A^k$ denote a top-down enumeration of those $\succeq_A$-classes where $I_A^1 := I_A(x)$. Put

$$A'(x) = \cup_{j \geq 1} \inf(I_A^j)$$

Note that $\{x\} \cup A'(x) \sim \{z\}$ for any $z \in \inf(I_A^1)$. We would like to claim $A'(x)$ is the desired aggregation set, but it may not be minimal. Consider the collection of all subsets, say $A''(x)$, of $A'(x)$ which have the property that $\{x\} \cup A''(x) \prec \{x\}$. Find a minimal (w.r.t. set inclusion) such set, call it $A_*(x)$. Since $u_{\min}(x, A) = u(z)$ for $z \in \inf(I_A^1)$ and $\inf(I_A^1) \succ \inf(I_A^2) \succ \cdots \succ \inf(I_A^k)$ (by lemma 2) we must have $\sup(A_*(x)) \cap \inf(I_A^1) \neq \emptyset$. It follows that $A_*(x)$ is an aggregation set for x with the property that $U(\sup(A_*(x))) = u_{\min}(x, A)$. Apply the conclusion of the preceding paragraph to the aggregation set $A_*(x)$ to obtain $U^{\mathcal{T}_x}(A) \leq u_{\min}(x, A)$.

Now consider the case where A contains a singleton temptation, i.e. $z \in B_t(x)$ with the property that $u_{\min}(x, A) = u(z)$. In this case, z is introduced at the root node of the tree $\mathcal{T}_x$, so that the u-minimum of each terminal node in this tree is trivially bounded above by $u(z)$ when z is on the menu. It follows that, for each $x \in A$, the value of the category function $U^{\mathcal{C}}(\cdot)$ on the x-tree $\mathcal{T}_x$ is bounded above by $u_{\min}(x, A)$. In other words, for each $x \in A$ w have $U^{\mathcal{T}_x}(A) \leq u_{\min}(x, A)$. For each $x \notin A$ we consider two cases, either (i) $A \cap \mathcal{C}_i = \emptyset, \forall \mathcal{C}_i \in \mathcal{L}(N_x)$ or (ii) $\mathcal{C}_i \cap A \neq \emptyset$ for some $\mathcal{C}_i \in \mathcal{L}(N_x)$. In the former case, the function $U^{\mathcal{T}_x}(A)$ does not enter into the domain of the maximization $U^{\mathcal{C}}(A) = \max U^{\mathcal{T}_x}(A)$. In the latter case, each node $\mathcal{C}_i$ with $\mathcal{C}_i \cap A \neq \emptyset$ contains a node $\mathcal{C}_i' \in \mathcal{L}(N_z)$ for each $z \in \mathcal{C}_i \cap A$. Note that $\min_{w \in \mathcal{C}_i \cap A} u(w) \leq \min_{w \in \mathcal{C}_i' \cap A} u(w) \leq U^{\mathcal{T}_z}(A)$, where $\mathcal{C}_i' \in \mathcal{L}(N_z)$ and $z \in \mathcal{C}_i \cap A$. Hence, we obtain $U^{\mathcal{T}_x}(A) \leq \max_{z \in A} U^{\mathcal{T}_z}(A) \leq \max_{z \in A} u_{\min}(z, A)$. Now recall that $U(A) = \max_{x \in A} u_{\min}(x, A)$ and putting together the bounds on $U^{\mathcal{T}_x}(A)$ for each tree $\mathcal{T}_x$ we obtain

$$U^{\mathcal{C}}(A) = \max U^{\mathcal{T}_x}(A) \leq \max_{x \in A} u_{\min}(x, A) = U(A).$$

Step 3: Check the inequality $U^{\mathcal{C}}(\cdot) \geq U(\cdot)$.
We now check the reverse inequality. First introduce some terminology. For notational brevity (for this step alone) we denote level i tree nodes as $x_j(i)$ (the j-th node in level i) and branches are denoted $x_j(i) \to x_k(i+1)$. Fix a menu A and some $x \in A$. Consider the x-tree $\mathcal{T}_x$ and consider the set of all directed paths along branches in the tree, $\Phi := \{\ell : \ell = \{x \to x_{i_1}(1) \to x_{i_2}(2) \to \cdots \to x_{i_{N_x}}(N_x)\}\}$. Let $\ell = (\ell_1, \ell_2, \ldots, \ell_{N_x})$ denote the specific nodes which lie along the path ℓ. We say that x is *unobstructed* in $\mathcal{T}_x$ by the menu A if there is a path $\ell \in \Phi$ such that $\ell_i \cap A = \{x\}, \forall \ell_i$. We claim that x is unobstructed in a menu A if and only if $x \in A^*$. Note that if we prove this, then it follows that the value of the category function $U^{\mathcal{C}}(A)$ on the tree $\mathcal{T}_x$ is $u(x)$, which proves $U^{\mathcal{C}}(A) \geq U(A)$. Hence, we reduce to proving that x is unobstructed in the x-tree $\mathcal{T}_x$ at the menu A if and only if $x \in A^*$.

For the "if" part of the claim assume that x is unobstructed. Then clearly the value of the category function on the tree $\mathcal{T}_x$ is $u(x)$. It follows that $x \in A^*$. Now for the reverse direction. Proceed by induction on the $(\succeq)|_X$-rank of x. That is, for the base step take $z \in \inf(X)$ and verify that: For any menu A with $z \in A$ and $z \in A^*$, the tree $\mathcal{T}_z$ is unobstructed by A. Now induct upwards. If x is the lowest ranked singleton in A^*, then the unobstruction claim is obvious. Thus, assume x is not the lowest ranked singleton in A^* and wlog that $x \sim A^*$ (since, if $\mathcal{T}_x$ is obstructed in A, it can only be obstructed by elements with $(\succeq)|_X$-rank strictly lower than x). Moreover, for the unobstruction claim we actually need only consider menus A of the form $A \equiv \{x\} \cup A'$ where $A' := \cup_{y:\{x\}\succ\{y\}} y$. Thus, we shall assume that A has this form.

Let $A_1(x), A_2(x), \ldots, A_k(x)$ be an enumeration of the aggregation sets of x. We claim that for each $A_i(x)$, there is some $z_i \in A_i(x)$ such that (i) the z_i-tree is unobstructed in A *and* (ii) $z_i \notin A$. Check this via contradiction. Assume $A_1(x)$ is such that for every $z \in A_1(x)$ either (i) z is obstructed in A, or (ii) $z \in A$. Then, when z is obstructed in A we have (by the induction hypothesis) $z \notin (A \cup \{z\})^*$. Let $\{z_1, \ldots, z_n\}$ be elements of $A_1(x)$ that are obstructed by A and let $\{z_{n+1}, z_{n+2}, \ldots, z_m\}$ be the elements of $A_1(x)$ that are unobstructed by A, but for which $z_i \in A$. Since we are alleging $\{z_1, z_2, \ldots, z_n\} \cup \{z_{n+1}, \ldots, z_m\} = A_1(x)$, we then obtain $(A_1(x) \cup A)^* \subseteq A^*$. It follows (by Strong Reduction) that $A^* \sim A_1(x) \cup A$. OTOH, by Strong Reduction again, $A \sim A^* \sim \{x\}$ - contradiction. Thus, for each $A_i(x)$ find z_i such that (i) z_i is unobstructed in A and (ii) $z_i \notin A$.

For what follows we will need to concatenate paths from different levels (in the x-tree $\mathcal{T}_x$). Let $\ell(i, j)$ denote a path connecting a node in level $\mathcal{L}(i)$ to a node in level $\mathcal{L}(j)$. Recall that the element x had k attraction sets. Let $z_1, \ldots, z_k$ be a list of unobstructed elements chosen respectively from $A_1(x), \ldots, A_k(x)$ (and such that $z_i \notin A$). Let $x(k)$ denote a node in level $\mathcal{L}(k)$ of the x-tree that contains $\{x, z_1, \ldots, z_k\}$, i.e. we sequentially attach $\{z_1\} \cup B_t(z_1), \{z_2\} \cup B_t(z_2), \ldots, \{z_k\} \cup B_t(z_k)$ to the initial node comprised of $\{x\} \cup B_t(x)$. Let $\ell(0, k)$ denote the unobstructed path from level 0 to level k that ends at the node $x(k)$. From the construction of the x-tree, we successively attach y_i-trees for some $y_1, y_2, \ldots, y_l$. Let $y_{i_1} = y_1 = z_{i_1}, y_{i_2}, \ldots, y_{i_k}$ be the subsequence of $\{y_1, y_2, \ldots, y_l\}$ where we first attach a z_i-tree in the x-tree construction algorithm. Let $\mathcal{L}(N_1), \ldots, \mathcal{L}(N_l)$ denote the terminal levels of the y_i-trees being attached to the nodes at level $\mathcal{L}(k)$. We inductively concatenate the unobstructed path $\ell(0, k)$ with an unobstructed path $\ell(k, k + N_1)$ and in turn with an unobstructed path $\ell(k + N_1, k + N_1 + N_2)$, and so on.

If $y_1 \in \{z_1, \ldots, z_k\}$ (call this the "unobstructed set"), there is an unobstructed path $\ell(k, k + N_1)$ from the node $x(k)$ to some node $x(k + N_1)$ in level $\mathcal{L}(k + N_1)$ of the partial tree obtained by concatenating the y_1-tree to the preceding levels

$\mathcal{L}(1), \mathcal{L}(2), \ldots, \mathcal{L}(k)$. Concatenate $\ell(0, k)$ to $\ell(k, k + N_1)$ to obtain an unobstructed path from level $\mathcal{L}(0)$ to level $\mathcal{L}(k + N_1)$. If $y_1 \in B_t(x)$, then $y_1 \notin A$ and y_1 must be unobstructed at A. The argument for this mimics the companion case for y_2 (resp. y_3) and is presented in more detail below. Now consider y_2. If $y_2 \notin x(k + N_1)$, then a copy of the node $x(k + N_1)$ is replicated on every level of the attached y_2-tree. Thus, take $\ell(k + N_1, k + N_1 + N_2)$ to be the path which concatenates the branches $\{x(k + N_1) := \mathcal{C}_0 \to \mathcal{C}_1\}, \{\mathcal{C}_1 \to \mathcal{C}_2 = x(k + N_1)\}, \ldots, \{\mathcal{C}_{N_2-1}(= x(k + N_1)) \to \mathcal{C}_{N_1}(= x(k + N_1))\}$. If $y_2 \in x(k + N_1)$, consider two cases. Either y_2 is in the unobstructed set or it is not. In the former case, replicate the argument for the y_1 case to extend the unobstructed path. If y_2 is not in the unobstructed set, then either (i) y_2 is first introduced at some level $\mathcal{L}(k + k_y)$ of the $y_1(= z_{i_1})$-tree or (ii) $y_2 \in B_t(x)$ or $B_t(z_1)$. Consider case (i). Let $x(k + k_y)$ be the unique predecessor node at this level for which there is an unobstructed path starting at $x(k + k_y)$ and terminating at $x(k + N_1)$. Since we attach a y_2-tree at some level of the construction of the y_1-tree, a portion of this path must pass through (unobstructed) an y_2-tree. Denote this path segment as $\ell(k_1, k_2)$ (where there are N_2 branches that comprise this segment). Note that the y_2-tree is embedded inside the y_1-tree, so that (by the recursiveness of the tree construction) the image of the same path in *any* embedded y_2-tree is unobstructed (w.r.t the menu A). Let $\ell(k + N_1, k + N_1 + N_2)$ be a copy of the path $\ell(k_1, k_2)$ with root node $x(k + N_1)$, and that passes through the y_2-tree. Consider the concatenation, $(\ell(0, k); \ell(k, k + N_1); \ell(k + N_1, k + N_1 + N_2))$ and note that this concatenation is unobstructed. Let the terminal node of this path be denoted $x(k + N_1 + N_2)$. This argument is summarized in the following schematic.
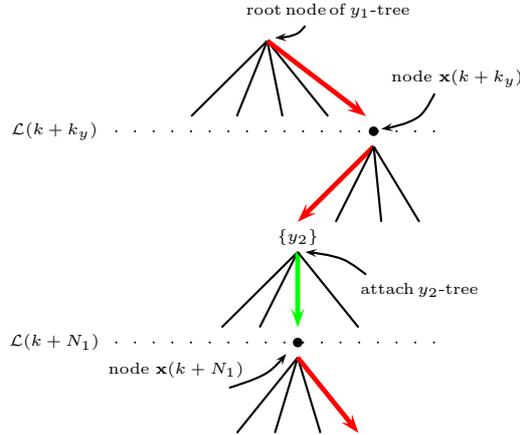

Figure 3: An unobstructed path through the y_1-tree which passes through the nested y_2-tree.

The green segment is the path $\ell(k_1, k_2)$ – this is an unobstructed path through the nested image of the y_2-tree inside the y_1-tree. This means that when we reach the stage of the reconstruction where we attach the y_2-tree we can use the same

path to extend the concatenation of an unobstructed path. Now consider case (ii). If $y_2 \in B_t(x)$, then we claim it must be unobstructed at A (in addition to $y_2 \notin A$). Else, $y_2 \notin (A \cup \{y_2\})^*$ (again, by the induction hypothesis). Hence, $(A \cup \{y_2\})^* \subseteq A^*$ which implies, by Strong Reduction, that $A^* \sim A \cup \{y_2\}$ and, in turn, $A \sim A \cup \{y_2\}$. OTOH, $\{x\} \sim A$ and $\{x\} \succ A \cup \{y_2\}$ – contradiction. Similarly, if $y_2 \in B_t(z_1)$ it must be unobstructed in A (in addition to $y_2 \notin A$). Else, an analogous argument shows that z_1 is obstructed in A. Hence, in either case we can find an unobstructed path going through the y_2-tree. As in case (i), we concatenate the initial path $(\ell(0, k); \ell(k, k + N_1))$ with an unobstructed path through the attached y_2-tree, call the latter $\ell(k_1 + N_1, k + N_1 + N_2)$. Note that the concatenation $(\ell(0, k); \ell(k, k + N_1); \ell(k + N_1, k + N_1 + N_2))$ is unobstructed. Next consider y_3. If y_3 is in the unobstructed set we proceed verbatim as above. Else, either (i) y_3 is first introduced along an unobstructed path in the y_i-tree for some $i \leq 2$ or (ii) $y_3 \in B_t(x), B_t(z_1)$, or $B_t(z_2)$ (the latter if $y_2 = z_2$). In case (i), a portion of one of the paths $\ell(k + N_1, k + N_1 + N_2)$ or $\ell(k, k + N_1)$ must pass unobstructed through the y_3-tree (at the point in the construction of the y_1 (resp. y_2) tree where the y_3 tree is attached). Let $\ell(k + N_1 + N_2, k + N_1 + N_2 + N_3)$ denote a replica of this path that goes through the y_3-tree. Continue the concatenation $(\ell(0, k); \ell(k, k + N_1); \ell(k + N_1, k + N_1 + N_2)); \ell(k + N_1 + N_2, k + N_1 + N_2 + N_3))$ to obtain an unobstructed partial path. Case (ii) is dealt with the same way as in the preceding argument (i.e. case (ii) for y_2). Inductively proceed to extend the concatenated path to obtain a sequence (put $k_i = k + \sum_{j=1}^{i} N_j$) $(\ell(0, k); \ell(k, k_1); \cdots; \ell(k_{l-1}, k_l))$. Note that this is a complete path in the x-tree and, moreover, it is unobstructed. It follows that the value of the category function $U^{\mathcal{C}}(\cdot)$ on the tree $\mathcal{T}_x$ is $u(x)$, proving that $U^{\mathcal{C}}(A) \geq U(A)$. This concludes the proof of the theorem. $\square$

Proof of Theorem 2. Menus (or, equivalently, nodes) will be bolded to denote the difference between an element and a set of elements. The argument follows two steps, (i) embedding and (ii) pruning. The first step shows that every category model $(u, \mathcal{C})$ that is sharp is a prolongation (of the form described in the theorem) of some sharp sub-category of the tree category. The second step shows that there is a unique sharp sub-category of the tree category.

Step 1: Embedding.
For each $x \in X$ consider $\Sigma'(x) = \{\mathcal{C}' \in \mathcal{C}_1 : x \in \mathcal{C}', x \in \sup(\mathcal{C}')\}$. If $\Sigma'(x) \neq \emptyset$, with labeling determined by the x-tree construction, let $A_1(x), \ldots, A_k(x)$ be the aggregation sets associated with x. We reconstruct a set of paths through the x-tree whose associated set of terminal nodes is contained in nodes in the set $\Sigma'(x)$. Introduce some notation. Let Φ be the set of all paths in the x-tree and for each $\ell \in \Phi$, let $\ell^{-1}(x_1(1))$ denote the set of all terminal nodes which have $\{x, x_1(1)\}$ as a predecessor node.[28] Similarly, for any (partial) path $\ell(1, n)$ from level 1 to level n, let $\ell^{-1}(\ell(1, n))$ denote the set of all terminal nodes whose associated paths through the tree all

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[28]Recall that when we put x in the node, this is shorthand for $\{x\} \cup B_t(x)$.

share the initial segment $\ell(1, n)$. Consider the aggregation set $A_1(x)$ and an element $x_1(1) \in A_1(x)$. We do not know yet that the sets in $\Sigma'(x)$ correspond to terminal nodes in the x-tree – this will be our conclusion. Nevertheless, we will apply the notation $\ell^{-1}(\ell(1, n))$ to the sets in $\Sigma'(x)$. The meaning is the following: For $x_1(1) \in A_1(x)$ we take $\ell^{-1}(x_1(1))$ to be the set of all elements (terminal nodes) in $\Sigma'(x)$ that contain the element $x_1(1)$. Similarly, for any $x_2(i) \in A_2(x)$ let $\ell^{-1}(x_1(1), x_2(i))$ denote all sets in $\Sigma'(x)$ which contain both $x_1(1)$ and $x_2(i)$. Note that, by representability, every node in $\Sigma'(x)$ is in $\ell^{-1}(x_1(i))$ for some $x_1(i) \in A_1(x)$ (there may be some nodes contained in $\ell^{-1}(x_1(i))$ for more than one i). Thus, we have a correspondence $\gamma_1 : A_1(x) \rightrightarrows \Sigma'(x)$ given by $x_1(j) \mapsto \ell^{-1}(x_1(j))$. Now iterate this process. For each $x_2(i)$ consider the set $\ell^{-1}(x_1(j), x_2(i))$ and note that, by representability, every node in $\Sigma'(x)$ lies in $\ell^{-1}(x_1(j), x_2(i))$ for some pair $x_1(j), x_2(i)$. Consider the correspondence $\gamma_2 : A_1(x) \times A_2(x) \rightrightarrows \Sigma'(x)$ given by $(x_1(i), x_2(j)) \mapsto \ell^{-1}(x_1(i), x_2(j))$. Inductively construct correspondences, $\gamma_n : \prod_{j=1}^{n} A_j(x) \rightrightarrows \Sigma'(x)$. Note that (after k levels) for the next step of the x-tree construction we attach an $x_i(j)$-tree for some $x_i(j) \in A_i(x)$. For any partial path $\ell(1, k)$ such that $\ell^{-1}(\ell(1, k)) \neq \emptyset$ we can extend via the same procedure as above to obtain an extension of the path through the $x_i(j)$-tree. Observe that each path that passes through a node that contains $x_i(j)$ extends by representability – there must be an element of any aggregation set associated to $x_i(j)$ contained in the node. Also note that if $x_i(j) \notin \mathbf{x}(k)$ for some $\mathbf{x}(k) \in \ell^{-1}(\ell(1, k))$ then this path is extended without branching until the terminal level of the $x_i(j)$-tree. Inductively proceeding we obtain a collection of paths in the x-tree.

Note that every node $\mathbf{x}' \in \Sigma'(x)$ has the property that $\mathbf{x}' \in \ell^{-1}(1, k), \forall k \leq k_\ell$ for some path in the x-tree of, say, length k_ℓ. Fix a node $\mathbf{x}' \in \Sigma'(x)$ and let $\{\ell_1, \ell_2, \ldots, \ell_n\}$ be an enumeration of paths in the x-tree (let k_i denote the length of path ℓ_i) such that $\mathbf{x}' \in \ell_i^{-1}(\ell(1, k)), \forall k \leq k_i$ and for all paths ℓ_i. Also let $\mathbf{x}(i)$ denote the terminal node (in the x-tree) of path ℓ_i. It follows that any node $\mathbf{x}' \in \Sigma'(x)$ contains some terminal nodes $\mathbf{x}(i)$ in the x-tree. We verify that (by retracting if necessary) any $\mathbf{x}'$ in a sharp model $(u, \mathcal{C})$ contains precisely one such $\mathbf{x}(i)$. Since $(u, \mathcal{C})$ is a sharp representation, for each set $\mathcal{C}_i \in \mathcal{C}$ one of the following must be true. Either

    i. There is a menu A for which $\arg\max_{\mathcal{C}_j \in \mathcal{C}} \min_{z \in \mathcal{C}_j \cap A} u(z) = \mathcal{C}_i$, or

    ii. $\mathcal{C}_i \nsubseteq \cup_{j \neq i} \mathcal{C}_j$.

In either case, we say that "the maximum occurs at x" on $\mathcal{C}_i$ if under case (i) the maximum of the function $U^{\mathcal{C}}(A)$ occurs at $x \in \mathcal{C}_i$ for some menu A, or if $x \notin \cup_{\mathbf{x}''(\neq \mathbf{x}') \in \mathcal{C}} \mathbf{x}''$ (in which case we can take the menu that supports this as a maximum to be $A = \{x\}$). Assume now that the maximum occurs at $x = \sup(\mathbf{x}')$. We claim that there is only one $\mathbf{x}(i) \subseteq \mathbf{x}'$. Else, if there is a distinct pair $(\mathbf{x}(i), \mathbf{x}(j))$

with $\mathbf{x}(i), \mathbf{x}(j) \subseteq \mathbf{x}'$ find $z \in \mathbf{x}(j) \backslash \mathbf{x}(i)$ (or $z \in \mathbf{x}(i) \backslash \mathbf{x}(j)$ if $\mathbf{x}(j) \subseteq \mathbf{x}(i)$). Consider the menu $A \cup \{z\}$. Since $z \notin \mathbf{x}(i)$ and $A \sim \{x\}$ (and $\mathbf{x}(i)$ is a terminal node in the tree category $(u, \mathcal{T})$ – which represents $\succeq$) we must have $A \cup \{z\} \sim \{x\}$. OTOH, $U^{\mathcal{C}}(A \cup \{z\}) < u(x)$ – contradicting the hypothesis that $(u, \mathcal{C})$ represents $\succeq$ as well. Hence, there is no such pair $(\mathbf{x}(i), \mathbf{x}(j))$.

Notice that exactly the same argument shows that the node $\mathbf{x}(i)$ coincides with $\mathbf{x}'$. Now assume the maximum occurs not at x, but for some $z_x \in \mathbf{x}'$ where $\{x\} \succ \{z_x\}$ (i.e. in case (i) holds, or $z_x \notin \cup_{\mathbf{x}'' \in \mathcal{C}} \mathbf{x}''$ in case (ii) holds). Take z_x to be the $(\succeq)|_X$-maximal such element. Replace $\mathbf{x}'$ in the category $\mathcal{C}$ with the lower bound order interval, $(-\infty, z_x] \cap \mathbf{x}' := \mathbf{x}_*$. Note that the category $\mathcal{C}' := \mathcal{C} \backslash \mathbf{x}' \cup \mathbf{x}_*$ (i.e. $\mathcal{C}'$ is obtained from $\mathcal{C}$ by deleting node $\mathbf{x}'$ and replacing it with $\mathbf{x}_*$, all other nodes are left intact) also represents $\succeq$. Now apply the preceding argument with z_x replacing x to find a unique terminal node $\mathbf{z}_{\mathbf{x}}(i)$ in the z_x-tree contained in $\mathbf{x}_* \subseteq \mathbf{x}'$. The preceding argument shows that $\mathbf{z}_{\mathbf{x}}(i) = (-\infty, z_x] \cap \mathbf{x}'$. Hence, $\mathbf{z}_{\mathbf{x}}(i)$ is a lower bound order interval in $\mathbf{x}'$.[29] This shows that every node $\mathbf{x}' \in \mathcal{C}$ contains a terminal node of the tree category $\mathcal{T}$ as a lower bound order interval. To complete the argument we need that no two nodes $\mathbf{x}', \mathbf{x}'' \in \mathcal{C}$ map to the same terminal node (under the mapping $\mathbf{x} \mapsto (-\infty, z_x] \cap \mathbf{x}$ described above) of the tree category as a lower bound order interval. However, this is a consequence of sharpness. If two nodes map to the same $(\infty, z_x] \cap \mathbf{x}$, then both cannot be relevant for the representation since for any menu where the maximum $U^{\mathcal{C}}(A)$ occurs on the node $\mathbf{x}$ (resp. $\mathbf{x}'$), it *must* occur on the common lower bound order interval $(-\infty, z_x] \cap \mathbf{x}$. Thus, not both of $\mathbf{x}, \mathbf{x}'$ can be relevant for the representation.

This concludes the argument for the embedding step. Let us summarize what has been shown so far. Starting with *any* sharp model $(u, \mathcal{C})$ we have shown that this model is a prolongation of a sharp model $(u, \mathcal{C}')$, where the latter model is a sub-category of the tree category $\mathcal{T}$. Moreover, each node, $\mathcal{C}'_i$, in $\mathcal{C}'$ is a lower bound order interval for a unique $\mathcal{C}_j$ in $\mathcal{C}$. We can further retract $(u, \mathcal{C}')$ to a (sharp) sub-model of the tree category with the property that for each node $\mathcal{C}'_i \in \mathcal{C}'$ the $(\succeq)|_X$-maximal element for which the maximum of the function $U^{\mathcal{C}}(\cdot)$ is attained on $\mathcal{C}'_i$ (for some menu A) is $\sup(\mathcal{C}'_i)$.[30] Formally, we take

$$\sup\{x_A \in \mathcal{C}'_i : \text{(i) } A \sim \{x_A\}, \text{(ii) } \mathcal{C}'_i = \arg\max_{\mathcal{C}'_j \in \mathcal{C}'} \min_{z \in \mathcal{C}'_j \cap A} u(z)\}$$

and reduce (by the preceding argument) to nodes $\mathcal{C}'_i$ such that $\sup(\mathcal{C}'_i) = \sup\{x_A : \text{(i) } A \sim \{x_A\}, \text{(ii) } \mathcal{C}'_i = \arg\max_{\mathcal{C}'_j \in \mathcal{C}'} \min_{z \in \mathcal{C}'_j \cap A} u(z)\}$. Put $x_{\mathcal{C}}$ equal to this common

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[29]The same argument for $\mathbf{x}'$ shows that the node $\mathbf{x}_*$ must contain some terminal node of the z_x-tree since the model $(u, \mathcal{C}')$ represents $\succeq$.

[30]This follows from identical reasoning as given in the preceding paragraph for the embedding argument.

element and note that $\mathcal{C}_i'$ *must* be a minimal (w.r.t. set inclusion) node in the $x_{\mathcal{C}}$-tree. Now turn to the next step.

Step 2: Pruning.
Fix the tree category representation of $(u, \mathcal{T})$ of $\succeq$. Take any two sharp sub-categories $(u, \mathcal{C}_1), (u, \mathcal{C}_2)$ which also represent $\succeq$ and have the property that for each $\mathcal{C}_1^i \in \mathcal{C}_1$ (resp. $\mathcal{C}_2^i \in \mathcal{C}_2$) the maximum of $U^{\mathcal{C}_1}(\cdot)$ (resp. $U^{\mathcal{C}_2}(\cdot)$) for some menu A is attained on $\sup(\mathcal{C}_1^i)$ (resp. $\sup(\mathcal{C}_2^i)$). By the preceding remarks, this implies that all nodes in $\mathcal{C}_1, \mathcal{C}_2$ are minimal nodes of the respective x-trees to which the nodes belong. We show that $\mathcal{C}_1 = \mathcal{C}_2$ by downwards induction on $\mathcal{T}_x$-trees. That is, we inductively show that all minimal nodes in the $\mathcal{T}_x$-tree that are present in $\mathcal{C}_1$ must also be in $\mathcal{C}_2$ and vice-versa. Start with the $(\succeq)|_X$-maximal x-tree. We claim that for maximal $\mathcal{T}_x$-trees *all* minimal terminal nodes are present in both $\mathcal{C}_1$ and $\mathcal{C}_2$. Towards contradiction, assume there is a minimal terminal node, say $\mathbf{x}_*$ that is not in the sub-category $(u, \mathcal{C}_1)$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be an enumeration of the minimal nodes in the $\mathcal{T}_x$-tree with $\mathbf{x}_1 \equiv \mathbf{x}_*$. For each $\mathbf{x}_i (\neq \mathbf{x}_1)$ pick some element $z_i \in \mathbf{x}_i \setminus \mathbf{x}_1$ (by minimality of $\mathbf{x}_*$). Consider the menu $A' := \{z_i : z_i \in \mathbf{x}_i\} \cup \{x\}$. Since the $\mathcal{T}_x$-tree is unobstructed at A' we must have $A' \sim \{x\}$. OTOH, consider the value of the function $U^{\mathcal{C}_1}(A')$ (where $U^{\mathcal{C}_1}(\cdot)$ denotes the menu utility generated by $(u, \mathcal{C}_1)$). Note that the only nodes in the category that contain x are the terminal nodes from the x-tree (by $(\succeq)|_X$-maximality). Since $\mathbf{x}_* \notin \mathcal{C}_1$ and $\{x\} \succ \{z_i\}$, where $z_i \in \mathbf{x}_i$, every node in $\mathcal{C}_1$ that contains the singleton x is obstructed by some z_i. It follows that $U^{\mathcal{C}_1}(A') < u(x)$, which contradicts representability since the $\mathcal{T}_x$-tree is unobstructed at the menu A'. It follows that all minimal terminal nodes for a $(\succeq)|_X$-maximal x-tree are present in the model $(u, \mathcal{C}_1)$ (and resp. $(u, \mathcal{C}_2)$, by the identical argument).

Similar reasoning shows that all minimal terminal nodes are present for any non-embedded $\mathcal{T}_x$-tree.[31] Accordingly break up the x-trees into two groups: group I is the set of non-embedded x-trees and group II is the set of embedded x-trees. Copying the preceding argument we check that $(u, \mathcal{C}_1)$ and $(u, \mathcal{C}_2)$ agree on all trees in group I, i.e. all nodes in $\mathcal{C}_1$ from group I trees are in $\mathcal{C}_2$ and vice-versa. Next, we induct downwards on $(\succeq)|_X$-rank to show that all nodes in $(u, \mathcal{C}_1)$ that come from group II trees are also in $(u, \mathcal{C}_2)$ and vice-versa. Enumerate group II trees via $\{x_1\} \succ \cdots \succ \{x_n\}$ and assume we have shown agreement of nodes in $\mathcal{C}_1$ and $\mathcal{C}_2$ on all $\mathcal{T}_{x_i}$-trees for $1 \leq i \leq k$. Consider the $\mathcal{T}_{x_{k+1}}$-tree and, towards contradiction, say that $\mathbf{x}_*$ is a node in $(u, \mathcal{C}_1)$ that is not in $(u, \mathcal{C}_2)$. Consider each $\mathcal{T}_{x_i}$-tree in which the $\mathcal{T}_{x_{k+1}}$-tree is embedded. Consider the nodes in this tree which contain the node $\mathbf{x}_*$ and which are present in $\mathcal{C}_2$. Formally denote this as:

$$\kappa_i(\mathbf{x}_*) := \{\mathbf{y}_i \in \mathcal{C}_2 : \mathbf{x}_* \subseteq \mathbf{y}_i\}.$$

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[31]By a "non-embedded" $\mathcal{T}_x$-tree we mean any $\mathcal{T}_x$-tree such that x does not appear in an aggregation set $A(x')$ for some $\{x'\} \succ \{x\}$.

We claim that $\kappa_i(\mathbf{x}_*)$ is non-empty for *some* x_i in which the $\mathcal{T}_{x_{k+1}}$-tree embeds. Proceed via contradiction. Consider all the minimal nodes of the x_{k+1}-tree and for each such node $\mathbf{x}_i \neq \mathbf{x}_*$ find $z_i \in \mathbf{x}_i \backslash \mathbf{x}_*$. Now note that, for any x_i-tree in which the x_{k+1}-tree embeds, any terminal node of this tree which contains x_{k+1} contains a node from the x_{k+1}-tree. If $\kappa_i(\mathbf{x}_*) = \emptyset$ for all x_i-trees in which the x_{k+1}-tree embeds, then each of these terminal nodes contains one of the nodes $\mathbf{x}_i \neq \mathbf{x}_*$ from the x_{k+1}-tree. Now consider the menu $A' := \{z_i \in \mathbf{x}_i : \mathbf{x}_i \neq \mathbf{x}_*\} \cup \{x\}$. Note that x_{k+1} is unobstructed at A'. However, the category $\mathcal{C}_1$ is obstructed since we are alleging that $\mathbf{x}_* \notin \mathcal{C}_1$ and that $\kappa_i(\mathbf{x}_*) = \emptyset, \forall i$ – contradiction. It follows that $\kappa_i(\mathbf{x}_*)$ is non-empty for some x_i-tree in which the x-tree embeds.

Now pick any $\mathbf{y}_i \in \kappa_i(\mathbf{x}_*)$. By the induction hypothesis, since $\mathbf{y}_i \in \mathcal{C}_2$ we also have $\mathbf{y}_i \in \mathcal{C}_1$. Since $\mathcal{C}_1$ is sharp, $\mathbf{x}_*$ cannot be a lower bound order interval in $\mathbf{y}_i$ (as deleting $\mathbf{x}_*$ from $\mathcal{C}_1$ would then yield a representation of the same menu preference, implying that $\mathbf{x}_*$ is not relevant for the representation – contradicting sharpness). Hence, for each such $\mathbf{y}_i$ we can find $z^i \in \mathbf{y}_i \backslash \mathbf{x}_*$ with $\{x_{k+1}\} \succ \{z^i\}$ and do this for every $\mathbf{y}_i \in \kappa_i(\mathbf{x}_*)$ and for every i with $\kappa_i(\mathbf{x}_*) \neq \emptyset$. Also find $z_i \in \mathbf{x}_i \backslash \mathbf{x}_*$ for each minimal node other than $\mathbf{x}_*$ of the x_{k+1}-tree. Let $A' := \{z_i : z_i \in \mathbf{x}_i \backslash \mathbf{x}_*\} \cup \{x_{k+1}\}$. Also put

$$A_i = \{z^i : \{x_{k+1}\} \succ \{z^i\}, z^i \in \mathbf{y}_i \backslash \mathbf{x}_*, \mathbf{y}_i \in \kappa_i(\mathbf{x}_*)\}.$$

Put these all together to construct the menu:

$$\hat{A} := A' \cup_i A_i.$$

Note that the x_{k+1}-tree is unobstructed at $\hat{A}$, so that $\hat{A} \sim \{x_{k+1}\}$. OTOH, computing the value of $U^{\mathcal{C}_2}(\hat{A})$ note that the embedded image of the node $\mathbf{x}_*$ in any x_i-tree is obstructed by $\hat{A}$ and, by construction, the nodes in $\mathcal{C}_2$ from the x_{k+1}-tree are also obstructed by $\hat{A}$. It follows that $U^{\mathcal{C}_2}(\hat{A}) < u(x_{k+1})$ – contradicting representability. Thus, $\mathbf{x}_* \in \mathcal{C}_2$, so that all nodes from the tree $\mathcal{T}_{x_{k+1}}$ that are in $\mathcal{C}_1$ are also in $\mathcal{C}_2$. By symmetry, all nodes from this tree that are in $\mathcal{C}_2$ are also in $\mathcal{C}_1$ – completing the inductive step. It follows that $\mathcal{C}_1 \equiv \mathcal{C}_2$. □

N.B. The argument shows that there is a unique, sharp sub-category of the tree category with the property that (i) all nodes are minimal and (ii) the highest value for which a strict maximum of the function $U^{\mathcal{C}}(\cdot)$ occurs on each of these nodes is the $(\succeq)|_X$-maximal singleton in the node. Note that it is easy enough to construct such a category from scratch. Namely, start with the tree category. Pass to the sub-category of all minimal nodes – which also yields a representation. Next, pass to *any* sharp sub-category. Note that the above argument already shows that all such sub-categories are prolongation of some $(u, \mathcal{C}_*)$. To find $\mathcal{C}_*$, go node by node and take $x_{\mathcal{C}_i}$ to be the maximum singleton for which the maximum is attained on node $\mathcal{C}_i$. If $x_{\mathcal{C}_i} \neq \sup(\mathcal{C}_i)$, then replace this node with $(-\infty, x_{\mathcal{C}_i}] \cap \mathcal{C}_i$. The same

arguments as presented above show that this must be a minimal terminal node of the $x_{\mathcal{C}_i}$-tree. Inductively proceed until we obtain a sharp sub-category consisting entirely of minimal nodes with $x_{\mathcal{C}_i} = \sup(\mathcal{C}_i)$ for all sets $\mathcal{C}_i$ in the sub-category.

6.2 Proofs for Sections 4.1-4.2

Proof of Lemma 1. That Strong CRW implies DSB is obvious. We check the converse. Consider the set $\Sigma(A) = \{x \in A : \{x\} \sim A\}$. We claim that there is some $x_A \in A$ such that $\forall A' \subseteq A$ with $x_A \in A'$ we have $A' \sim A$. Otherwise, for each x such that $\{x\} \sim A$ there is a subset $A(x) \subseteq A$ with $x \in A(x)$ and $A \not\sim A(x)$. Put $\hat{A} := A \setminus \cup_{x \in \Sigma(A)} A(x)$ and note that $A = \cup_{x \in \Sigma(A)} A(x) \cup \hat{A}$. By iterative application of DSB, $A \sim \hat{A}$ which implies (by iterative application of DSB again) that $\exists x \in \hat{A}$ such that $\{x\} \sim A$. On the other hand, $x \in A(x)$ and $A(x) \cap \hat{A} = \emptyset$ – contradiction. $\square$

Proof of Observation 1. Fix (u, v) and for each x consider the set $\mathcal{C}_x := \{x\} \cup \{y : v(y) > v(x), u(x) > u(y)\}$. Let $\mathcal{C} \equiv \{\mathcal{C}_x\}_{x \in X}$. Let $U^{\mathcal{C}}$ denote the menu utility generated by $(u, \mathcal{C})$ and let U denote the menu utility generated by (u, v) via the Strotz formula. Fix any menu A and let $x \in A_v$ be a u-maximal element. Note that $U(A) = u(x)$. Moreover, $\mathcal{C}_x \cap A = \{x\}$ so that $U^{\mathcal{C}}(A) \geq u(x)$. For the reverse inequality, take any $z \in A$ with $u(z) > u(x)$. Then, $z \notin A_v$ so that there is some $y \in A_v$ with $v(y) > v(z)$. Since $u(z) > u(x)$ and x is u-maximal in A_v we have $u(z) > u(x) \geq u(y)$, so that $y \in \mathcal{C}_z$. Hence, $U^{\mathcal{C}}(A) \leq u(x)$. It follows that $U^{\mathcal{C}}(A) = U(A)$. $\square$

Proof of Observation 2. We first check that $U(A_x) \geq \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}$. Consider the GP kernel, $u(z) - (\max_{w \in A_x} v(w) - v(z))$, and break into two cases, (i) the maximum occurs on x and (ii) the maximum occurs away from x, say at some $y \neq x$ (so that $u(y) + v(y) > u(x) + v(x)$). Consider this latter case first and note that $U(A_x) \leq u(y)$. If $U(A_x) = u(y)$, then we must have $u(y) \leq u(x)$ (by definition of A_x) so that $y \in A_1(x)$. It follows that $U(A_x) \geq \min_{z \in A_1(x)} u(z)$. Next, note that $U(A_x) < u(y)$ implies that $\exists z_y \in A_x$ with $u(y) + v(y) \geq u(z_y) + v(z_y)$ and $v(z_y) = \max_{z \in A_x} v(z) > v(y)$. Since $u(y) + v(y) > u(x) + v(x)$, we obtain that $u(y) - (v(z_y) - v(y)) > u(x) - (v(z_y) - v(x))$. Note that $u(z_y) \leq u(x)$, else $U(A_x) = u(y) - (v(z_y) - v(y)) \geq u(z_y) > u(x)$ (the inequality since y is chosen from A_x). But $U(A_x) \leq u(x)$ by definition of A_x, a contradiction. If $z_y \in A_1(x)$, then $u(z_y) + v(z_y) > u(x) + v(x)$ so that $U(\{x, z_y\}) = u(z_y)$. Moreover, $u(y) - (v(z_y) - v(y)) \geq u(z_y)$. Hence, $\{z_y\} \sim \{x, z_y\} \preceq A_x$. It follows that, in this case as well, $U(A_x) \geq \min_{z \in A_1(x)} u(z)$. To finish this case, we now claim that $z_y \notin A_2(x)$. Else, x is chosen from $\{x, z_y\}$ so that $U(\{x, z_y\}) = u(x) - (v(z_y) - v(x)) < u(y) - (v(z_y) - v(y)) = U(A_x)$. On the other hand, $\{x, z_y\} \succeq A_x$ by definition of the set A_x – a contradiction. Now consider the case where the maximum occurs on

x, i.e. x is chosen in A_x. Note that $U(A_x) \leq u(x)$. If $u(x) = U(A_x)$, then we must have $v(x) = \max_{z \in A_2(x)} v(z)$. Hence, $U(A_x) = u(x) - (\max_{z \in A_2(x)} v(z) - v(x)) \geq \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}$. If $U(A_x) < u(x)$, then find z_x with $u(x) + v(x) \geq u(z_x) + v(z_x)$ and $v(z_x) = \max_{z \in A_x} v(z) > v(x)$. Note that $z_x \in A_2(x)$ so that $U(A_x) = u(x) - (v(z_x) - v(x)) \geq u(x) - (\max_{z \in A_2(x)} v(z) - v(x))$. Hence, $U(A_x) \geq \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}$ in this case as well. We now check the reverse inequality: $U(A_x) \leq \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}$. For each $z \in A_1(x)$ note that $\{x, z\} \sim \{z\}$. Hence, $\{z\} \sim \{x, z\} \succeq A_x$, as $U(A_x) = \min\{U(A') : A' \subseteq A, x \in A'\}$. Similarly, $U(A_x) \leq U(A_2(x))$. Moreover, note that $U(A_2(x)) = u(x) - (\max_{z \in A_2(x)} v(z) - v(x))$ as x is chosen from $A_2(x)$ (by definition of $A_2(x)$). Hence, $U(A_x) \leq \min\{\min_{z \in A_1(x)} u(z), u(x) - (\max_{z \in A_2(x)} v(z) - v(x))\}$. $\qquad\square$

Proof of Observation 3. Take any set A' and note that for any $\mathcal{C}_i \in \mathcal{C}(x)$ we have $\min_{x \in \mathcal{C}_i \cap A'} u(x) \geq \min_{x \in \mathcal{C}_i \cap A} u(x)$. It follows that

$$U(A') = \max_{\mathcal{C}_i \in \mathcal{C}} \min_{x \in \mathcal{C}_i \cap A'} u(x) \geq \max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{x \in \mathcal{C}_i \cap A'} u(x)$$
$$\geq \max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{x \in \mathcal{C}_i \cap A} u(x).$$

Hence, it suffices to find a set, A_x, for which the value $U(A_x)$ is attained on $\max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{x \in \mathcal{C}_i \cap A} u(x)$. Begin with $\overline{A} := \cup_{\mathcal{C}_i \in \mathcal{C}(x)} (\mathcal{C}_i \cap A)$. If the value is attained on some $\mathcal{C}_i \in \mathcal{C}(x)$, then this set is our A_x and we are done. Else, if the maximum is attained on some set $\mathcal{C}_i \notin \mathcal{C}(x)$, then let z_0 be an element on which the value is attained. Note that we must have $u(z_0) > \min_{x \in \mathcal{C}_i \cap \overline{A}} u(x), \forall \mathcal{C}_i \in \mathcal{C}(x)$. Consider $\overline{A} \backslash z_0$. The value $U(\overline{A} \backslash z_0)$ must either (i) be attained on $\mathcal{C}_i \in \mathcal{C}(x)$ or (ii) attained on some $\mathcal{C}_i \notin \mathcal{C}(x)$. Let z_1 be the element that attains the maximum and note that $u(z_1) > \min_{x \in \mathcal{C}_i \cap (\overline{A} \backslash z_0)} u(x), \forall \mathcal{C}_i \in \mathcal{C}(x)$. Hence, we may remove z_1 from the menu without affecting the value of $\min_{x \in \mathcal{C}_i \cap (\overline{A} \backslash z_0)} u(x), \forall \mathcal{C}_i \in \mathcal{C}(x)$. Consider $\overline{A} \backslash \{z_0, z_1\}$ and repeat the argument. Eventually we must reach a set $\overline{A} \backslash \{z_0, \ldots, z_k\} =: A_x$ where the maximum occurs on some $\mathcal{C}_i \in \mathcal{C}(x)$. Note that $\min_{x \in \mathcal{C}_i \cap A_x} u(x) = \min_{x \in \mathcal{C}_i \cap A} u(x), \forall \mathcal{C}_i \in \mathcal{C}(x)$. It follows that $u_{\min}(x, A) = \max_{\mathcal{C}_i \in \mathcal{C}(x)} \min_{x \in \mathcal{C}_i \cap A} u(x)$. $\qquad\square$

Proof of Observation 4. We check by induction on the cardinality of A that $u(x, A) \leq u_{\min}(x, A)$. The inequality trivially holds for singleton A, thus let $|A| > 1$. By hypothesis, $u(x, A) \leq u(x, A \backslash y), \forall y \in A \backslash x$. By the induction hypothesis, $u(x, A \backslash y) \leq u_{\min}(x, A \backslash y)$. Let A_x be an element of $\arg\min\{U(A') : A' \subseteq A, x \in A\}$ and note that if $\exists y \in A$ such that $A_x \subseteq A \backslash y$, then $u_{\min}(x, A) = u_{\min}(x, A \backslash y)$ – which implies $u(x, A) \leq u_{\min}(x, A)$. If there is no such y, then it must be the case that $\arg\min\{U(A') : A' \subseteq A, x \in A'\} = \{A\}$. In this case we obtain, on the one hand, $u_{\min}(x, A) = U(A)$. On the other hand, $U(A) = \max_{x \in A} u(x, A)$, so that $u(x, A) \leq u_{\min}(x, A)$. $\qquad\square$

41

6.3 Proofs for Section 4.3

Proof of Proposition 1. Put $\Lambda(\succeq) := (u, C)$ and check this pair satisfies A1-3. A1 and A3 are clearly satisfied from construction of the pair (u, C). For A2, let us first check the equality: $A^* = A^*_{(u,C)}$. That is, the subjective feasible set computed relative to $\succeq$ agrees with the feasible set derived from the ex post observables. The left-to-right containment is obvious from definition. Now check right-to-left. Let $x \in A^*_{(u,C)}$ and note that, by definition, if $A' \subseteq A$ s.t. $x \in A'$, then $u(y) \geq u(x), \forall y \in C(A')$ where $C(A') := \arg\max_{x \in (A')^*} u(x)$. If, towards contradiction, $x \notin A^*$, then there is some subset $A' \subseteq A$ where $x \in A'$ yet $\{x\} \succ A'$. This implies that $x \notin (A')^*$. Moreover, by Axiom 1 we know that $A' \sim \{x_{A'}\}$ for some $x_{A'} \in (A')^*$. Since, by definition, $A' \succeq \{y\}, \forall y \in (A')^*$ we obtain: $\{x\} \succ A' \sim \{x_{A'}\} \succeq y, \forall y \in (A')^*$. Hence, $u(x) > u(y), \forall y \in (A')^*$, implying that $x \notin A^*_{(u,C)}$ – a contradiction. This shows that $\Lambda(\succeq)$ satisfies axioms A1-A3.

For the converse, let $(u, C) \in \Sigma(X)$. We want to find an order on menus $\succeq \in \Sigma_{\mathbf{P}(X)}$ such that $\Lambda(\succeq) = (u, C)$. For each menu A, pick any $x_A \in C(A)$ and define a utility on menus via the formula $U(A) := u(x_A)$. By A1, this is a well-defined function on menus. Let us check that it satisfies Axiom 1 (CRW) and Axiom 2 (Strong Reduction). Consider $A' \subseteq A$ with $x_A \in A'$. Since $x_A \in C(A)$, this implies $C(A) \cap A' \neq \emptyset$. By A3, this implies that $u(x) \geq u(y), \forall x \in C(A'), \forall y \in C(A)$. Hence, $U(A') = u(x_{A'}) \geq u(x_A) = U(A)$, implying Axiom 1 (CRW). To check Axiom 2 (Strong Reduction) consider two sets, $A^*_{(u,C)}$ and A^*. The latter is the subjective feasible set computed w.r.t. the induced menu preference $\succeq$ (defined via $U(A) = u(x_A)$). The former is the subjective feasible set defined using the observables (u, C). Notice that if (u, C) satisfies $A1 - A3$, then these two sets are identical. Axiom 2 (Strong Reduction) then follows immediately. Hence, $\succeq \in \Sigma_{\mathbf{P}(X)}$. To finish, we verify that $\Lambda(\cdot)$ is a bijection. Since we have already checked surjectivity it suffices to verify that Λ is injective. Towards contradiction, say that $\succeq, \succeq' \in \Sigma(X)$ map to the same (u, C). This implies $(\succeq)|_X = (\succeq')|_X$, i.e. the singleton ranking is the same. Note that if $\succeq \neq \succeq'$, then there must be some menu A such that $A^*_{\succeq} \neq A^*_{\succeq'}$.[32] This implies that there is some x with $-$ wlog $- x \in A^*_{\succeq} \backslash A^*_{\succeq'}$. Let $I_A(x)$ denote the $\succeq'_A$-indifference class of x and consider the menu, $D := \{z \in A : (i) \{x\} \succ \{z\}, (ii) x \succeq'_A z\} \cup \{x\}$. By Axiom 2 (Strong Reduction) we know that $x \notin D^*_{\succeq'}$. OTOH, since $x \in A^*_{\succeq}$, we must have $x \in D^*_{\succeq}$. Since $x = \sup(D^*_{\succeq})$ it follows that $x \in C(D^*_{\succeq})$, contradicting the fact that $C(D^*_{\succeq}) = C(D^*_{\succeq'})$. Hence, there is no such pair $(A^*_{\succeq}, A^*_{\succeq'})$, implying that $\succeq = \succeq'$. □

Proof of Proposition 2. Given a filter Θ and a normative ranking u we take $C(\cdot)$ to be $C(A) := \arg\max_{x \in \Theta(A)} u(x)$. We verify that when the filter satisfies Sen's

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[32] $A^*_{\succeq}$ denotes the feasible set computed under $\succeq$ and $A^*_{\succeq'}$ denotes the feasible set computed under $\succeq'$.

α, the two restricted β conditions, and the worst elements are feasible restriction, then (i) the filter equals the subjective feasible set defined by the pair (u, C) and (ii) the induced pair (u, C) satisfies A1-A3. These two facts imply, using the preceding proposition, that the filter comes from a category model. To this end, we first show that $\Theta(A)$ coincides with the feasible set A^* induced by (u, C). Note that $\Theta(A) \subseteq A^*$ follows from Sen's α. We verify that $A^* \subseteq \Theta(A)$. Towards contradiction, assume that $x \in A^*$, yet $x \notin \Theta(A)$. Note that by feasibility we have $\max_{z \in \Theta(A)} u(z) \geq u(x)$. Hence, consider the (allegedly non-empty) subset $\{z \in \Theta(A) : u(z) \geq u(x)\} := A_1$ and put $A_2 := \{x\} \cup \{z \in A : u(x) > u(z)\}$. Note that $A := (A \backslash A_2) \cup A_2$ and $\min_{z \in A \backslash A_2} u(z) \geq u(x)$ and, moreover, $A_1 \subseteq A \backslash A_2$ so that the latter menu is non-empty. By subjective feasibility we must have $x \in \Theta(A_2)$. Now note that, by WCF, $\arg\min_{z \in A} u(z) \subseteq \Theta(A) \cap A_2$. By the restricted β, I we obtain $x \in \Theta(A)$ – contradiction. It follows that $\Theta(A) = A^*, \forall A$. This implies that $C(A) := \arg\max_{x \in \Theta(A)} u(x) = \arg\max_{x \in A^*} u(x)$. In other words, if we start out with a pair (u, Θ) where Θ satisfies α, the restricted β condition, and WCF, and consider the induced choice correspondence $C(\cdot)$, then we can recover this choice correspondence by maximizing $u(\cdot)$ on the set A^*.

We now check that the axioms $A1 - A3$ hold for the (induced) pair (u, C). Note that axiom $A1$ follows from definition of C and $A3$ follows from Sen's α. We now check $A2$. Put $A^* \subseteq A' \subseteq A$ and assume $x \in \Theta(A')$. If x is u-minimal in A', then since u-minimal elements of A are in A^* (by WCF):

$$\min_{y \in A} u(y) \leq \min_{y \in A'} u(y) \leq \min_{y \in A^*} u(y) \leq \min_{y \in A} u(y).$$

The first inequality since $A \supseteq A'$, the second since $A' \supseteq A^*$, and the third by WCF again. Hence, by WCF we have $x \in \Theta(A)$. If $x \notin \min_{y \in A'} u(y)$, find $z \in \min_{y \in A'} u(y)$. By WCF $z \in \Theta(A')$ and by the preceding argument this implies $z \in \Theta(A)$. By the restricted Sen's β, II this then implies $x \in \Theta(A)$ as well (since $\Theta(A) = A^*$ by the previous argument). Hence, $A^* = \Theta(A) = \Theta(A') = (A')^*$, implying that $C(A) = C(A')$.

$\square$

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