Negishi's method for stochastic OLG models

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Motivation

- OLG models are important tools in macro and public finance
- Unfortunately, humans typically live for 80 years ...
- A large number of generations creates two problems:
  - High-dimensional state space to work without
  - Large system of functional equations to solve
- Can we reduce the computational burden by choosing the 'right' state variable?

Motivation

- It is often convenient to characterize equilibrium allocations as solutions to a social planner's problem.
  As a state variable, one can then use the *instantaneous Negishi weights*.
- In OLG models, however, using Negishi weights seems odd, because the Negishi weights of the newborn are not known, while their portfolios are given.
- We show that it is nevertheless advantageous to use Negishi weights as state variable in OLG. It can reduce the computational burden of solving high-dimensional OLG models!

This Paper

- Consider an OLG exchange economy with a Lucas tree:
  - Characterize equilibria that are recursive in Negishi weights
  - Discuss existence of such recursive equilibria
  - Show that computation becomes much simpler if Negishi weights are used as state variable
- First, consider the case of complete financial markets.
- Second, assume that markets may be incomplete in two ways:
  1. not all future labor-endowments may be pledged, and
  2. available assets do not allow for full spanning.

Related Literature

- Instantaneous Negishi weights as the 'right' state-variable: Negishi (1960); Cuoco and He (1994,2001); Dumas and Lyasoff (2012); Chien and Lustig (2011); Chien, Cole and Lustig (2011)
- Existence of recursive equilibria in OLG models: Duffie et al. (1994); Kubler and Polemarchakis (2004); Citanna and Siconolfi (2010)
- Computation in OLG models: Krueger and Kubler (2004); Storesletten, Telmer and Yaron (2004); Feng, Miao, Peralta-Alva and Santos (2013)

The Physical Economy

- Time is discrete and infinite,
  shock process $\{s_t\}$ is Markov with finite support $\mathcal{S}$,
  histories are denoted by $\sigma = s^t = (s_0, \ldots, s_t) \in \Sigma$
- Each period H agents are born who live for A periods,
  identified by $(a, h) \in \mathcal{A} := \{1, \ldots, A\} \times \mathcal{H}$, $\mathcal{H} := \{1, \ldots, H\}$
- There are L perishable commodities,
  individual endowments $\omega_{a,h}(s_t) \in \mathbb{R}_+^L$
  depend on age, type and current shock.
- Lucas tree in unit net supply, pays dividends $d(s_t) \in \mathbb{R}_+^L$.
- Preferences of type h born at s^t are given by

$$U_{s^t,h}(x) = u_{1,h}\left(x(s^t)\right) + \sum_{a=1}^{A-1} \sum_{s^{t+a} \succeq s^t} \delta_{a,h}(s^{t+a}) u_{a+1,h}\left(x(s^{t+a})\right),$$

  with discount factors $\delta_{a,h}(s^t) > 0$ satisfying:

$$\delta_{1,h}(s^t) = 1, \delta_{a,h}(s^{t+a}) = \delta_{a-1,h}(s^{t+a-1})\delta_{a,h}(s_{t+a-1}, s_{t+a}).$$

Complete Markets and Negishi Weights

- With complete markets and a Lucas tree, competitive equilibrium is Pareto efficient (e.g. Demange (2002)).

$\Rightarrow$ There exist Pareto weights $\{\eta_{s^t,h}\}_{s^t \in \Sigma, h \in \mathcal{H}}$ such that

$$(x_{s^t,h})_{s^t \in \Sigma, h \in \mathcal{H}} = \arg\max \sum_{s^t \in \Sigma, h \in \mathcal{H}} \eta_{s^t,h} U_{s^t,h}(x_{s^t,h})$$

$$\text{s.t.} \sum_{s^t \in \Sigma, h \in \mathcal{H}} x_{s^t,h}(\sigma) \leq \bar{\omega}(\sigma) \text{ for all } \sigma \in \Sigma,$$

where $\bar{\omega}(\sigma) = \sum_{(a,h) \in \mathcal{A}} \omega_{a,h}(\sigma) + d(\sigma)$.

- Using Pareto weights, define instantaneous Negishi weights, $\lambda(s^t) = (\lambda_{a,h}(s^t))_{(a,h) \in \mathcal{A}}$ by

$$\lambda_{a,h}(s^t) = \eta_{s^{t-a+1},h} \delta_{a,h}(s^t).$$

Individual consumption is then given by $X : \mathcal{S} \times \mathbb{R}_+^{AH} \to \mathbb{R}^{AHL}$:

$$X(s, \lambda) \equiv \arg\max_{x \in \mathbb{R}_+^{AHL}} \sum_{(a,h) \in \mathcal{A}} \lambda_{a,h} u_{a,h}(x_{a,h}) \text{ s.t. } \sum_{(a,h) \in \mathcal{A}} x_{a,h} \leq \bar{\omega}(s).$$

Recursive Equilibrium and Negishi Weights

- Want to describe a recursive equilibrium by a transition function:
$$\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}.$$
- Given functions X and Λ, define financial wealth by

$$\begin{aligned} W_{A,h}(s,\lambda) &= D_x u_{a,h}(X_{A,h}(s,\lambda)) \cdot (X_{A,h}(s,\lambda) - \omega_{A,h}(s)), \\ W_{a,h}(s,\lambda) &= D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - \omega_{a,h}(s)) + \\ & \sum_{s'} \delta_{a+1,h}(s,s') W_{a+1,h}(s', \Lambda(s,s',\lambda)), \text{ for } a < A. \end{aligned}$$

Definition

A recursive equilibrium is a function $\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}$ that satisfies for all $s, s' \in \mathcal{S}$ and all $\lambda \in \mathbb{R}^{AH}_{++}$,
$$W_{1,h}(s', \Lambda(s,s',\lambda)) = 0 \text{ for all } h \in \mathcal{H} \text{ and}$$
$$\Lambda_{a,h}(s,s',\lambda) = \delta_{a,h}(s,s')\lambda_{a-1,h} \text{ for all } (a,h) \in \mathcal{A}_{-1},$$
where $\mathcal{A}_{-1} := \mathcal{A} \setminus \{(a,h) : a = 1\}$.

Existence of Generalized Markov Equilibria

- In the spirit of Duffie et al. (1994), we first consider equilibria that are Markov on a larger state space.
- In particular, we include financial wealth W in the state space.
- Building on Kubler and Polemarchakis (2004), we prove existence of such generalized Markov equilibria.

Existence of Recursive Equilibria

In order for a generalized Markov equilibrium to be a recursive equilibrium, we need single-valuedness of the correspondence $W(s, \lambda)$

- Guaranteed by gross substitutes assumption
- Always satisfied for $A = 2$. Interestingly, for $A = 2$, Kubler and Polemarchakis (2004) present a counterexamples to existence of recursive equilibrium on the natural state space.

$\Rightarrow$ Choice of state variable can make a difference with respect to existence of recursive equilibria. Standard recursive methods assume that there is a function from W to λ. But it is 'more likely' that there is a function from λ to W ...

Existence of Minimal Recursive Equilibria

- As the newborn did not make any decisions in the past, do we need their Negishi weights as a state variable?
- A recursive equilibrium is said to be *minimal*, if there exists a function

$$\ell(s, (\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}}) : \mathcal{S} \times \mathbb{R}_{++}^{(A-1)H} \to \mathbb{R}_{++}^{H},$$

  such that
  $\Lambda_{1,h}(s, s', \lambda) = \ell_h \left(s', (\Lambda_{a,h}(s, s', \lambda))_{(a,h)\in\mathcal{A}_{-1}}\right)$ for all $h \in \mathcal{H}$.
  We can then define Λ and W on $\mathbb{R}_{++}^{(A-1)H}$ rather than $\mathbb{R}_{++}^{AH}$.
- We show that the gross substitutes assumption is also sufficient for existence of a minimal recursive equilibrium.

Recap: Equilibrium Concepts and Existence

	Generalized Markov Equilibrium	Recursive Equilibrium	Minimal Recursive Equilibrium
Endogenous State Variable	$\{\lambda_{a,h}, W_{a,h}\}_{(a,h)\in\mathcal{A}}$	$\{\lambda_{a,h}\}_{(a,h)\in\mathcal{A}}$	$\{\lambda_{a,h}\}_{(a,h)\in\mathcal{A}_{-1}}$
Dimension of Endogenous State Space	$2 \cdot A \cdot H$	$A \cdot H$	$(A-1) \cdot H$
Existence	✓	gross substitutes or $A = 2$	gross substitutes

Computation of Recursive Equilibria

- Is it simpler to compute recursive equilibria if Negishi weights rather than wealth is used as the state variable?
- To identify the advantages of the Negishi approach, we look at a time iteration collocation algorithm, as in e.g. Kubler and Krueger (2004)
- However, many advantages remain if other methods are used, e.g. Rios-Rull (1996)

Computing (Minimal) Recursive Equilibria with Collocation

1. Choose collocation grid $\mathcal{G} = \{\lambda_{-1}^{1}, \ldots, \lambda_{-1}^{G}\} \subset \mathbb{R}_{++}^{(A-1)H}$, guess $\hat{W}^0 : \mathcal{S} \times \mathbb{R}_{+}^{(A-1)H} \to \mathbb{R}^{(A-1)H}$, and set $n = 0$.

2. For each $(s, \lambda_{-1}^{i}) \in \mathcal{S} \times \mathcal{G}$, compute $\hat{\ell}^{n+1}(s, \lambda_{-1}^{i})$ solving

$$D_x u_{1,h}(X_{1,h}(s, \lambda^i(s))) \cdot (X_{1,h}(s, \lambda^i(s)) - \omega_{1,h}) + \sum_{s'} \delta_{2,h}(s,s') \hat{W}_{2,h}^{n}(s', \lambda_{-1}^{i}(s')) = 0, \;\; h \in \mathcal{H},$$

   where $\lambda^i(s) = (\hat{\ell}^{n+1}(s, \lambda_{-1}^{i}), \lambda_{-1}^{i})$,
   and $\lambda_{-1}^{i}(s') = (\delta_{a,h}(s,s') \lambda_{a-1,h}^{i}(s))_{(a,h) \in \mathcal{A}_{-T}}$.

3. For each $(s, \lambda_{-1}^{i}) \in \mathcal{S} \times \mathcal{G}$, compute for all $(a,h) \in \mathcal{A}_{-1}$

$$\hat{W}_{a,h}^{n+1}(s, \lambda_{-1}^{i}) = D_x u_{a,h}(X_{a,h}(s, \lambda^i(s))) \cdot (X_{a,h}(s, \lambda^i(s)) - \omega_{a,h}) + \sum_{s'} \delta_{a+1,h}(s,s') \hat{W}_{a+1,h}^{n}(s', \lambda_{-1}^{i}(s')),$$ where $\hat{W}_{A+1,h}^{n+1}(s, \lambda_{-1}^{i}) := 0$.

4. For each s, interpolate $\{\hat{W}^{n+1}(s, \lambda_{-1}^{i})\}_{\lambda^i \in \mathcal{G}}$ to get $\hat{W}^{n+1}(s, .)$. Check error, then increase n by 1 and go to 2 or finish.

Computational Advantages of the Negishi-Approach I

1. We have to solve only systems of H equations (as opposed to $(A-1)HS$ FOCs to get portfolio choices)
2. Given (s, s'), policy functions may be defined over the $(A-1)H-1$ dimensional unit simplex, as policies are homogeneous of degree zero in Negishi weights. Thus the shape of the state space is simple and known (in contrast to the case of cash-at-hand)

Computational Advantages of the Negishi-Approach II

3. Approximation errors have a nice interpretation. The error

$$\sup_{h\in\mathcal{H},s\in\mathcal{S},\lambda_{-1}\in\mathbb{R}^{(A-1)H}} \left\| \frac{\hat{W}_{1,h}(s',\hat{\ell}(s,\lambda_{-1}))}{\frac{\partial u_{1,h}(X_{a,h}(s,\lambda))}{\partial x_1}} \right\|$$

is the maximal transfer necessary to turn the computed allocation into an equilibrium allocation.

4. The computational complexity barely increases with the number of goods, L, as only the computation of $X(s,\lambda)$ and of $D_x u_{a,h}(X(s,\lambda))$ depends on L
(as opposed to solving for spot-prices and allocations simultaneously with portfolios and asset-prices)

The General Model

In the paper, we consider a very general setup,
that allows markets to be incomplete in two ways:

1. not all future labor-endowments may be pledged, and
2. available assets do not allow for full spanning.

In this talk, we discuss two special cases:

1. Borrowing constraints with full spanning, and
2. Limited spanning without constraints.

Borrowing Constraints with Full Spanning

- Maintain assumption that there is a full set of Arrow securities.
- Assume that agents can borrow against their Lucas tree holding, but only against part of their future labor endowments.
- Gottardi and Kubler (2012) and Chien and Lustig (2011) examine a similar version of this model, but assume infinitely lived agents

Borrowing constraints

- Individual endowments

$$\omega_{a,h}(s_t) = e_{a,h}(s_t) + f_{a,h}(s_t) \text{ for all } s_t.$$

are composed of

1. intangible endowments, e, that cannot be pledge and
2. tangible endowments, f, that can be pledged.

- Because of the borrowing constraints,

$$\delta_{a,h}(s_t) D_x u_{a,h}(X(s, \lambda_{a,h}(s_t))$$

are not collinear to market prices, but

$$\lambda_{a,h}(s_t) D_x u_{a,h}(X(s, \lambda_{a,h}(s_t))$$

are, as these are identical across all agents alive at s_t and at least one agent (a, h) is unconstrained and thus satisfies

$$\lambda_{a+1,h}(s_{t+1}) = \delta_{a+1,h}(s_t, s_{t+1}) \lambda_{a,h}(s^t).$$

Defining financial wealth and tangible wealth

- Therefore, modify the definition of financial wealth:

$$
\begin{aligned}
W_{A,h}(s,\lambda) &= \lambda_{A,h} D_x u_{a,h}(X_{A,h}(s,\lambda)) \cdot (X_{A,h}(s,\lambda) - \omega_{A,h}(s)), \\
W_{a,h}(s,\lambda) &= \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - \omega_{a,h}(s)) + \\
&\quad \sum_{s'} W_{a+1,h}(s', \Lambda(s,s',\lambda)), \text{ for } a < A.
\end{aligned}
$$

- Given functions X and Λ, define the tangible wealth by

$$
\begin{aligned}
V_{A,h}(s,\lambda) &= \lambda_{A,h} D_x u_h(X_{A,h}(s,\lambda)) \cdot (X_{A,h}(s,\lambda) - e_{A,h}(s)) \\
V_{a,h}(s,\lambda) &= \lambda_{a,h} D_x u_h(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - e_{a,h}(s)) + \\
&\quad \sum_{s'} V_{a+1,h}(s', \Lambda(s,s',\lambda)), \text{ for } a < A.
\end{aligned}
$$

- The borrowing constraint demands that tangible wealth is non-negative

Recursive Equilibrium and Negishi Weights

Definition

A recursive equilibrium is a function $\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}$ that satisfies for all $s, s' \in \mathcal{S}$ and all $\lambda \in \mathbb{R}^{AH}_{++}$,

$$W_{1,h}(s', \Lambda(s, s', \lambda)) = 0 \text{ for all } h \in \mathcal{H} \text{ and}$$

$$\Lambda_{a,h}(s, s', \lambda) = \delta_{a,h}\lambda_{a-1,h} + \eta_{a,h} \text{ with } \eta_{a,h} \geq 0,$$

$$V_{a,h}(s', \Lambda(s, s', \lambda)) \geq 0, \quad V_{a,h}(s', \Lambda(s, s', \lambda))\eta_{a,h} = 0$$

for all $(a, h) \in \mathcal{A}_{-1}$.

Existence of Recursive Equilibria

Taking care of the V-function, we can get the same results as for the complete markets case:

- Existence of generalized Markov equilibria (which now include also V as state) can be proved
- Gross substitutes assumption guarantees existence of recursive equilibrium and also of minimal recursive equilibrium

Computation of (Minimal) Recursive Equilibria

- We now have to guess $\hat{V}^0 : \mathcal{S} \times \mathbb{R}_+^{(A-1)H} \to \mathbb{R}^{(A-1)H}$ and solve from $\hat{V}^n$ for $\hat{V}^{n+1}$ which involves the following step:
  2. Given $\hat{V}^n$, for each $s \in \mathcal{S}$ and each $\lambda_{-1}^i \in \mathcal{G}$, compute $\hat{\eta}^n(s, \lambda_{-1}^i)$ as a solution to the non-linear complementarity problem

$$\hat{V}^n(s, \lambda_{-1}^i + \hat{\eta}^n(s, \lambda_{-1}^i)) \geq 0, \quad \hat{\eta}^n(s, \lambda_{-1}^i)) \geq 0$$
$$\hat{V}_{a,h}^n(s, \lambda_{-1}^i + \hat{\eta}^n(s, \lambda_{-1}^i))\hat{\eta}_{a,h}^n(s, \lambda_{-1}^i) = 0 \text{ for all } (a, h) \in \mathcal{A}_{-1}$$

  Interpolate $\{\hat{\eta}^n(s, \lambda_{-1}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{\eta}^n(s, .)$.

- In contrast to complete markets, the size of the system now depends on A; yet it does not depend on S (as opposed to the case of the natural state space)

Computational Advantages of the Negishi-Approach

1. Size of nonlinear systems to solve does not depend on S
2. Given (s, s'), policy functions may be defined over the $(A-1)H-1$ dimensional unit simplex
3. Approximation errors have a nice interpretation
4. The computational complexity barely increases with the number of goods

Limited spanning without constraints

- Can we also use instantaneous Negishi weights in models without a full set of state-contingent claims?
- Can still characterize equilibrium using W-functions; however, we need spanning conditions that involve portfolios
- Therefore, nonlinear equation systems are not smaller than with wealth as state variable
- The other computational advantages remain though!

Conclusion

Short Summary:

- We study equilibria of OLG models that are recursive in Negishi weights
- Show existence under gross substitutes assumption for the case of borrowing constraints with full spanning
- Show computational advantages of the Negishi approach:
  - When markets are complete, there are huge advantages
  - Relaxing the complete markets assumption, substantial advantages remain

Outlook:

- Combine Negishi's approach with adaptive sparse grid methods to approximate large scale OLG models