

Intergenerational Bargaining and Inter Vivos Transfers*

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Abstract

Altruistic dynasty models in which family members care about the consumption or utility of other family members can account for the existence of inter vivos transfers and intended bequests between family members, but they struggle to match the quantitative properties of data regarding these gifts. For example, dynasty models predict that families should undo any exogenous transfer between family members (such as Social Security) with a countervailing endogenous transfer, but this is not observed empirically. Here we consider a simple generalization of the two-period overlapping generations model in which both the parent and the child care about the other family member's consumption, and the net transfer between them is determined by cooperative bargaining. While this model nests altruistic dynasty models, more generally we find that transfers only occur when the parent to child wealth ratio is above or below certain thresholds. If the parent and child have reciprocal feelings towards each other, transfers will only happen when one party is extremely poor relative to the other.

KEYWORDS: overlapping generations model, inter vivos transfers, Nash bargaining, optimal consumption, dynamic game

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We study to what extent augmenting standard overlapping generations (OLG) models with inter vivos transfers alters the key insights we have learned from such models. OLG models are workhorse models for studying a wide array of topics, such as the wealth distribution (Cagetti and DeNardi 2006, 2009, DeNardi 2004, Yang 2005), entrepreneurship formation (Cagetti and DeNardi 2006), welfare effects of the Great Recession (Glover, Heathcote, Krueger, and Rios-Rull 2011), and life cycle dynamics of consumption and saving (Bullard and Feigenbaum 2007), among others. A common feature of the canonical OLG model is that, apart from bequest motives, households in each generation are modelled largely as maximizing only their own welfare. Although bequests are a significant component of intergenerational transfers, they are by no means the only transfers. In particular, bequests are typically one-way transfers—from parents to children—and only take place at the end of the life cycle. Meanwhile, inter vivos intergenerational transfers between living members of a family are quite common. For example, Kaplan (2010) documented that young adults under financial pressure often move back to live with their parents so they may economize resources spent on rent, utility and food. Moreover, ample anecdotal evidence suggests that elderly parents often get financial and in-kind transfers from their adult children.

With the seminal contribution of Barro (1974), students of OLG models came to the realization that finite lives may not have any implications for the capitalization of future tax liabilities if generations are linked by a chain of operational intergenerational transfers. In a similar spirit, we ask what happens if households of different generations living concurrently in the same family intrinsically care about each other’s welfare and engage in inter vivos transfers. To what extent would an otherwise standard OLG model behave like a model with infinitely-lived consumers and how would the key results derived from typical OLG models change? This is a quantitative question, the answer to which depends crucially on the strength of intergenerational welfare linkage and the way in which transfers are determined.

A key innovation of our model is that we allow inter vivos transfers to be state-dependent and to happen in both directions—parents transferring resources to children and children transferring resources to parents. Accordingly, the size and direction of the transfer is determined in a Nash-bargaining setup. We show that earlier intergenerational transfer models, such as Altonji, Hayashi, and Kotlikoff (1997), can be nested in our Nash-bargaining model with specific parameterizations of the bargaining power between generations.

To gain insight into the mechanism of such an intra-family intergenerational Nash-bargaining model, we focus here on a stylized model in which each generation lives for two periods, coexisting first with their parents and then in the second period of life with their children. We present analytical results for the cases where either the parent or the child has all bargaining power and show that such models are observationally equivalent to those of the altruistic model, which in turn implies results observationally equivalent to those of models with infinitely-lived consumers such as Aiyagari (1994). We next turn to the more interesting case where both the parent and child have some bargaining power. If neither generation has complete bargaining power, the model will generally behave differently from models with infinitely-lived consumers unless parents and children both assign the same relative weight to utility from each generation. Also, Nash-bargaining implies that transfers only occur when the parent-child wealth-ratio is far enough from unity that a transfer will increase the welfare of both the transfer recipient and provider. One surprising result is how the family responds to exogenous transfers from the child to the parent as described in Altonji, Hayashi, and Kotlikoff (1997). In an altruism model like Barro (1974), the allocation of consumption to the parent and child only depends on the total wealth of the family, so the family should respond to such an exogenous transfer by undoing it. In the present model with bargaining, we find that the apportionment of wealth between the parent and child matters. If the disparity of wealth between the parent and child is small, they will not undo the exogenous transfer. When the parent to child wealth ratio exceeds a threshold level, however, the parent will provide a countervailing transfer that more than undoes the exogenous transfer from the child to the parent.

1 The Model

We consider a two-period overlapping generations model. Every household has one child household.¹ When young, a household coexists with its parents. When old, a household coexists with its children. Each chain of households defines a dynasty with intergenerational links similar to Yang (2005).

A household in dynasty i born at time t will maximize

$$u(c_{t,t}^i) + v_p(c_{t,t-1}^i) + \beta[u(c_{t+1,t}^i) + v_c(c_{t+1,t+1}^i)] \quad (1)$$

subject to

$$\begin{aligned} c_{t,t}^i + b_{t+1}^i &= y_{t,t}^i + \tau_t^i \\ c_{t+1,t}^i + \tau_{t+1}^i &= y_{t+1,t}^i + Rb_{t+1}^i \\ c_{t,t-1}^i + \tau_t^i &= x_{t,t-1}^i \equiv y_{t,t-1}^i + Rb_t^i \\ c_{t+1,t+1}^i + b_{t+2}^i &= y_{t+1,t+1}^i + \tau_{t+1}^i, \end{aligned}$$

where $c_{t,s}^i$ is consumption at time t of the household in dynasty i born at time s ; $y_{t,s}^i$ is the income at time t of the household in dynasty i born at time s ; b_t^i is the saving of the household in dynasty i born at $t-1$; τ_t^i is the transfer from the parent in dynasty i to its child (which could be negative) at time t ; and $R > 0$ is the gross interest rate. The household's own period utility function is $u(c)$, the utility that it gets from its parent or child's contemporaneous consumption is $v(c)$, and $\beta > 0$ is the discount factor.

Let us suppose that we have consumption functions $c_{t,s}^i(\tau_{t'}^i)$ expressed as a function of the transfer at time t' , the details of which will be specified below. Then τ_t^i is determined by Nash bargaining with a threat point of no transfer, which is always feasible if the expected transfer next period is not too large.² Thus we maximize

$$\begin{aligned} L^N(\tau_t^i) &= [u(c_{t,t-1}^i(\tau_t^i)) + v_c(c_{t,t}^i(\tau_t^i)) - u(c_{t,t-1}^i(0)) - v_c(c_{t,t}^i(0))]^\theta \\ &\quad \times [u(c_{t,t}^i(\tau_t^i)) + v_p(c_{t,t-1}^i(\tau_t^i)) + \beta[u(c_{t+1,t}^i(\tau_t^i)) + v_c(c_{t+1,t+1}^i(\tau_t^i))] \\ &\quad - u(c_{t,t}^i(0)) - v_p(c_{t,t-1}^i(0)) - \beta[u(c_{t+1,t}^i(0)) + v_c(c_{t+1,t+1}^i(0))]]^{1-\theta}, \end{aligned} \quad (2)$$

where θ is the relative bargaining power of the parent and $1-\theta$ is the relative bargaining power of the child.

To define an equilibrium we will have to specify a bargaining mechanism to determine the transfer between each contemporaneous pair of generations.

We assume income is nonstochastic with $y_{t,s} = y_{t-s}$ for $t = s, s+1$.

The policy functions for a young household in dynasty i at period t will be a function of the parent's wealth $x_{t,t-1}^i$. We will need to solve for consumption functions $c_{t,t}^i(x_{t,t-1}^i)$, $c_{t,t-1}^i(x_{t,t-1}^i)$, a bond demand function $b_{t+1}^i(x_{t,t-1}^i)$, an expected transfer function $\tilde{\tau}_{t+1}^i(x_{t,t-1}^i)$ and a transfer function $\tau_t^i(x_{t,t-1}^i)$. A young household solves its problem with the expectation that the transfer when old will be governed by $\tilde{\tau}_{t+1}^i$. In a rational-expectations equilibrium we must have

$$\tilde{\tau}_{t+1}^i(x_{t,t-1}^i) = \tau_{t+1}^i(y_1 + Rb_{t+1}^i(x_{t,t-1}^i)) \quad (3)$$

for all $x_{t,t-1}^i$.

¹We may wish to generalize the model for the case where the number of children is stochastic as in Feigenbaum (2011). However, that will require an agent-based model.

²We have to adjust the threat point for the case where the expected transfer is larger than the child's present value of wealth. This will only happen when the parent's wealth is very large. See below.

We compute an equilibrium according to the following algorithm. Given $\tilde{\tau}_{t+1}^i(x_{t,t-1}^i)$, for each possible state $x_{t,t-1}^i$ we solve the young household's problem for $\tilde{c}_{t,t}^i(\tau_t^i)$ and $\tilde{b}_{t+1}^i(\tau_t^i)$. Then $\tau_t^i(x_{t,t-1}^i)$ is determined to solve the bargaining problem between the young and old household at t . The next iteration of $\tilde{\tau}_{t+1}^i(x_{t,t-1}^i)$ is then computed according to (3).

Let $u(c)$ be CRRA with risk aversion $\gamma > 0$ and let $v_i(c) = \chi_i u(c)$, where $\chi_i \geq 0$ for $i = p, c$.

We also assume that $y_{t,s}^i = y_{t-s}$. Given τ_t^i and τ_{t+1}^i , the household born at t will maximize

$$u(c_{t,t}) + \beta u(c_{t+1,t})$$

subject to

$$c_{t,t} + b_{t+1} = y_0 + \tau_t \tag{4}$$

$$c_{t+1,t} + \tau_{t+1} = y_1 + Rb_{t+1} \tag{5}$$

Thus

$$\begin{aligned} L(b_{t+1}|\tau_t, \tau_{t+1}) &= u(y_0 + \tau_t - b_{t+1}) + \beta u(y_1 + Rb_{t+1} - \tau_{t+1}) \\ \frac{dL(b_{t+1}|\tau_t, \tau_{t+1})}{db_{t+1}} &= -(y_0 + \tau_t - b_{t+1})^{-\gamma} + \beta R(y_1 + Rb_{t+1} - \tau_{t+1})^{-\gamma} = 0 \\ y_0 + \tau_t - b_{t+1} &= (\beta R)^{-1/\gamma} (y_1 + Rb_{t+1} - \tau_{t+1}) \end{aligned}$$

Define

$$\begin{aligned} \phi &= (\beta R^{1-\gamma})^{-1/\gamma} \tag{6} \\ y_0 + \tau_t - b_{t+1} &= \phi \left(\frac{y_1 - \tau_{t+1}}{R} + b_{t+1} \right) \\ (1 + \phi)b_{t+1} &= y_0 + \tau_t - \frac{\phi}{R}(y_1 - \tau_{t+1}) \end{aligned}$$

Thus the bond demand as a function of the transfers is

$$b_{t+1}(\tau_t, \tau_{t+1}) = \frac{1}{1 + \phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - \tau_{t+1}) \right]. \tag{7}$$

The consumption functions are

$$\begin{aligned} c_{t,t}(\tau_t, \tau_{t+1}) &= y_0 + \tau_t - b_{t+1}(\tau_t, \tau_{t+1}) = y_0 + \tau_t - \frac{1}{1 + \phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - \tau_{t+1}) \right] \\ c_{t,t}(\tau_t, \tau_{t+1}) &= \frac{\phi}{1 + \phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \tag{8} \end{aligned}$$

$$\begin{aligned} c_{t+1,t} &= y_1 - \tau_{t+1} + Rb_{t+1} = y_1 - \tau_{t+1} + \frac{R}{1 + \phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - \tau_{t+1}) \right] \\ &= \frac{1}{1 + \phi} [R(y_0 + \tau_t) + (1 + \phi - \phi)(y_1 - \tau_{t+1})] \\ c_{t+1,t}(\tau_t, \tau_{t+1}) &= \frac{R}{1 + \phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \tag{9} \end{aligned}$$

Having solved for the consumption functions, let us guess at τ_{t+1} and compute τ_t that maximizes (2). In a symmetric Nash equilibrium, we will have $\tau_t = \tau_{t+1} = \tau$. The parent at t will have cash on hand $x_{t,t-1}$. Holding τ_{t+1} and τ_{t+2} fixed, the utility gain of the parent will be

$$U_t^p(\tau_t) = u(x_{t,t-1} - \tau_t) - (x_{t,t-1}) + \chi_c \left(u \left(\frac{\phi}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \right) - u \left(\frac{\phi}{1+\phi} \left[y_0 + \frac{y_1 - \tau_{t+1}}{R} \right] \right) \right)$$

$$U_t^p(\tau_t) = u(x_{t,t-1} - \tau_t) - (x_{t,t-1}) + \left(\frac{\phi}{1+\phi} \right)^{1-\gamma} \chi_c \left[u \left(y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right) - u \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \right]. \quad (10)$$

The utility gain of the child will be

$$\begin{aligned} U_t^c(\tau_t) &= u \left(\frac{\phi}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \right) + \beta u \left(\frac{R}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \right) \\ &\quad + \chi_p u(x_{t,t-1} - \tau_t) + \beta \chi_c u \left(\frac{\phi}{1+\phi} \left[y_0 + \tau_{t+1} + \frac{y_1 - \tau_{t+2}}{R} \right] \right) \\ &\quad - u \left(\frac{\phi}{1+\phi} \left[y_0 + \frac{y_1 - \tau_{t+1}}{R} \right] \right) - \beta u \left(\frac{R}{1+\phi} \left[y_0 + \frac{y_1 - \tau_{t+1}}{R} \right] \right) \\ &\quad - \chi_p u(x_{t,t-1}) - \beta \chi_c u \left(\frac{\phi}{1+\phi} \left[y_0 + \tau_{t+1} + \frac{y_1 - \tau_{t+2}}{R} \right] \right) \end{aligned}$$

Note that

$$\begin{aligned} u \left(\frac{\phi}{1+\phi} x \right) + \beta u \left(\frac{R}{1+\phi} x \right) &= \frac{1}{1-\gamma} \left(\frac{x}{1+\phi} \right)^{1-\gamma} [\phi^{1-\gamma} + \beta R^{1-\gamma}] \\ &= \frac{1}{1-\gamma} \left(\frac{x}{1+\phi} \right)^{1-\gamma} [\phi^{1-\gamma} + \phi^{-\gamma}] \\ &= u(x) \frac{\phi^{-\gamma}}{(1+\phi)^{-\gamma}} = (1+\phi^{-1})^\gamma u(x). \end{aligned}$$

Thus the utility gain of the child simplifies to

$$U_t^c(\tau_t) = (1+\phi^{-1})^\gamma \left[u \left(y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right) - u \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \right] + \chi_p [u(x_{t,t-1} - \tau_t) - u(x_{t,t-1})] \quad (11)$$

So both the parent and the child's utility are linear combinations of

$$u \left(y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right) - u \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right)$$

and

$$u(x_{t,t-1} - \tau_t) - u(x_{t,t-1}).$$

They only differ in the relative weights.

An abstract formulation of this problem is

$$U_1 = A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2 + \tau) - u(x_2)] \quad (12)$$

$$U_2 = A_{21}[u(x_1 - \tau) - u(x_1)] + A_{22}[u(x_2 + \tau) - u(x_2)]. \quad (13)$$

For the Nash bargaining problem, we wish to choose τ to maximize

$$V = \theta \ln U_1 + (1 - \theta) \ln U_2 \quad (14)$$

$$\frac{dV}{d\tau} = \frac{\theta}{U_1} [-A_{11}(x_1 - \tau)^{-\gamma} + A_{12}(x_2 + \tau)^{-\gamma}] + \frac{1 - \theta}{U_2} [-A_{21}(x_1 - \tau)^{-\gamma} + A_{22}(x_2 + \tau)^{-\gamma}] = 0$$

Thus we need

$$\begin{aligned} \left(\frac{\theta A_{11}}{U_1} + \frac{(1 - \theta) A_{21}}{U_2} \right) (x_1 - \tau)^{-\gamma} &= \left(\frac{\theta A_{12}}{U_1} + \frac{(1 - \theta) A_{22}}{U_2} \right) (x_2 + \tau)^{-\gamma} \\ (\theta A_{11} U_2 + (1 - \theta) A_{21} U_1) (x_1 - \tau)^{-\gamma} &= (\theta A_{12} U_2 + (1 - \theta) A_{22} U_1) (x_2 + \tau)^{-\gamma} \end{aligned} \quad (15)$$

$$\begin{aligned} \theta A_{11} U_2 + (1 - \theta) A_{21} U_1 &= (\theta A_{11} A_{21} + (1 - \theta) A_{21} A_{11}) [u(x_1 - \tau) - u(x_1)] \\ &\quad + (\theta A_{11} A_{22} + (1 - \theta) A_{21} A_{12}) [u(x_2 + \tau) - u(x_2)] \\ &= A_{11} A_{21} [u(x_1 - \tau) - u(x_1)] \\ &\quad + (\theta A_{11} A_{22} + (1 - \theta) A_{12} A_{21}) [u(x_2 + \tau) - u(x_2)] \end{aligned}$$

$$\begin{aligned} \theta A_{12} U_2 + (1 - \theta) A_{22} U_1 &= ((1 - \theta) A_{22} A_{11} + \theta A_{12} A_{21}) [u(x_1 - \tau) - u(x_1)] \\ &\quad + (\theta A_{12} A_{22} + (1 - \theta) A_{22} A_{12}) [u(x_2 + \tau) - u(x_2)] \\ &= ((1 - \theta) A_{22} A_{11} + \theta A_{12} A_{21}) [u(x_1 - \tau) - u(x_1)] \\ &\quad + A_{12} A_{22} [u(x_2 + \tau) - u(x_2)] \end{aligned}$$

This gives us

$$\begin{aligned} &(A_{11} A_{21} [u(x_1 - \tau) - u(x_1)] + (\theta A_{11} A_{22} + (1 - \theta) A_{21} A_{12}) [u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ &= (((1 - \theta) A_{22} A_{11} + \theta A_{12} A_{21}) [u(x_1 - \tau) - u(x_1)] + A_{12} A_{22} [u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \end{aligned} \quad (16)$$

This does not have an analytic solution for general θ .

If $\theta = 0$ or 1, so either the parent or the child makes all the decisions, it will simplify to the usual solution. If $\theta = 0$,

$$\begin{aligned} &(A_{11} A_{21} [u(x_1 - \tau) - u(x_1)] + A_{21} A_{12} [u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ &= (A_{22} A_{11} [u(x_1 - \tau) - u(x_1)] + A_{12} A_{22} [u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \end{aligned}$$

$$\begin{aligned} &A_{21} (A_{11} [u(x_1 - \tau) - u(x_1)] + A_{12} [u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ &= A_{22} (A_{11} [u(x_1 - \tau) - u(x_1)] + A_{12} [u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \end{aligned}$$

$$A_{21} (x_1 - \tau)^{-\gamma} = A_{22} (x_2 + \tau)^{-\gamma}$$

$$A_{21}^{-1/\gamma} (x_1 - \tau) = A_{22}^{-1/\gamma} (x_2 + \tau)$$

$$(A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}) \tau = A_{21}^{-1/\gamma} x_1 - A_{22}^{-1/\gamma} x_2$$

Thus when Player 2 has all the bargaining power,

$$\tau = \frac{A_{21}^{-1/\gamma} x_1 - A_{22}^{-1/\gamma} x_2}{A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}}$$

$$c_1 = D_1(x_1 - \tau) = D_1 \left(x_1 - \frac{A_{21}^{-1/\gamma} x_1 - A_{22}^{-1/\gamma} x_2}{A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}} \right) = D_1 \frac{A_{22}^{-1/\gamma}}{A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}} (x_1 + x_2) \quad (17)$$

$$c_2 = D_2(x_2 + \tau) = D_2 \left(x_2 + \frac{A_{21}^{-1/\gamma} x_1 - A_{22}^{-1/\gamma} x_2}{A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}} \right) = D_2 \frac{A_{21}^{-1/\gamma}}{A_{21}^{-1/\gamma} + A_{22}^{-1/\gamma}} (x_1 + x_2) \quad (18)$$

In this case

$$\frac{c_2}{c_1} = \frac{D_2}{D_1} \left(\frac{A_{22}}{A_{21}} \right)^{1/\gamma}.$$

Conversely, if $\theta = 1$, the bargaining condition becomes

$$\begin{aligned} & (A_{11}A_{21}[u(x_1 - \tau) - u(x_1)] + A_{11}A_{22}[u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ = & (A_{12}A_{21}[u(x_1 - \tau) - u(x_1)] + A_{12}A_{22}[u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \\ & A_{11}(x_1 - \tau)^{-\gamma} = A_{12}(x_2 + \tau)^{-\gamma} \\ & A_{11}^{-1/\gamma}(x_1 - \tau) = A_{12}^{-1/\gamma}(x_2 + \tau) \\ & \tau = \frac{A_{11}^{-1/\gamma} x_1 - A_{12}^{-1/\gamma} x_2}{A_{11}^{-1/\gamma} + A_{12}^{-1/\gamma}} \end{aligned}$$

Thus

$$c_1 = D_1(x_1 - \tau) = D_1 \frac{A_{12}^{-1/\gamma}}{A_{11}^{-1/\gamma} + A_{12}^{-1/\gamma}} (x_1 + x_2) \quad (19)$$

$$c_2 = D_2(x_2 + \tau) = D_2 \frac{A_{11}^{-1/\gamma}}{A_{11}^{-1/\gamma} + A_{12}^{-1/\gamma}} (x_1 + x_2) \quad (20)$$

These are the same as (17)-(18) except now it is the weights of player 1 that matter rather than the weights of player 2.

In this case

$$\frac{c_2}{c_1} = \frac{D_2}{D_1} \left(\frac{A_{12}}{A_{11}} \right)^{1/\gamma}.$$

There is a third analytic case when $A_{21} = kA_{11}$ and $A_{22} = kA_{12}$, so the proportional weights are the same:

$$\begin{aligned} & (kA_{11}^2[u(x_1 - \tau) - u(x_1)] + (\theta kA_{11}A_{12} + (1 - \theta)kA_{11}A_{12})[u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ = & (((1 - \theta)kA_{12}A_{11} + \theta kA_{12}A_{11})[u(x_1 - \tau) - u(x_1)] + kA_{12}^2[u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \end{aligned}$$

$$\begin{aligned} & A_{11} (A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma} \\ = & A_{12} (A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma} \end{aligned}$$

$$A_{11}(x_1 - \tau)^{-\gamma} = A_{12}(x_2 + \tau)^{-\gamma}$$

This just gives us (19)-(20) again, which in this case are equivalent to (17)-(18).

These preliminary results suggest that Barro's (1974) results that an overlapping-generations model can be equivalent to an infinite-horizon representative-agent model may not require pure altruism if either the parent or the child has all the bargaining power, a situation that may be relevant to many developing nations like Uzbekistan.

1.1 Paterfamilias Model

Let us focus on the case where $\theta = 1$ so the parent has all the bargaining power to see how closely this is to a Barro model. Since the parent has absolute power over the child, we call this the paterfamilias model. Let us define

$$\xi = \left(\left(\frac{1}{1+\phi} \right)^{1-\gamma} \chi_c \right)^{-1/\gamma}, \quad (21)$$

so

$$\begin{aligned} A_{11} &= 1 \\ A_{12}^{-1/\gamma} &= \xi. \end{aligned}$$

We also have

$$\begin{aligned} D_1 &= 1 \\ D_2 &= \frac{\phi}{1+\phi} \\ \frac{c_{t,t}}{c_{t,t-1}} &= \frac{\phi}{1+\phi} \frac{1}{\xi} \end{aligned}$$

Then the optimal transfer must satisfy

$$\tau_t = \frac{x_{t,t-1} - \xi \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right)}{1 + \xi}. \quad (22)$$

Let us assume the transfer is a linear function:

$$\tau(x) = Ax + B \quad (23)$$

$$\begin{aligned} x_{t+1,t} &= y_1 + Rb_{t+1} = y_1 + \frac{R}{1+\phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - Ax_{t+1,t} - B) \right] \\ x_{t+1,t} &= y_1 + \frac{R}{1+\phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - B) \right] + \frac{\phi}{1+\phi} Ax_{t+1,t} \\ x_{t+1,t} &= \frac{1}{1 - \frac{\phi}{1+\phi}A} \left[y_1 + \frac{R}{1+\phi} \left[y_0 + \tau_t - \frac{\phi}{R}(y_1 - B) \right] \right] \\ x_{t+1,t} &= \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{R}{1+\phi} \left[\frac{1+\phi}{R} y_1 + y_0 + \tau_t - \frac{\phi}{R}(y_1 - B) \right] \\ x_{t+1,t} &= \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{R}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] \\ x_{t+1,t} &= \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{R}{1+\phi} \left[y_0 + Ax_{t,t-1} + B + \frac{y_1 + \phi B}{R} \right] \end{aligned} \quad (24)$$

$$\tau_{t+1} = \frac{R}{\phi} \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A\phi}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] + B \quad (25)$$

Substituting this into (22)

$$Ax_{t,t-1} + B = \frac{x_{t,t-1} - \xi \left(y_0 + \frac{y_1 - \frac{R}{\phi} \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A\phi}{1+\phi} [y_0 + Ax_{t,t-1} + B + \frac{y_1 + \phi B}{R}] - B \right)}{1 + \xi}$$

$$A = \frac{1 + \xi \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A^2}{1+\phi}}{1 + \xi} \quad (26)$$

$$A(1 + \xi) - 1 = \xi \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A^2}{1 + \phi} \quad (27)$$

$$\begin{aligned} [A(1 + \xi) - 1] \left(1 - \frac{\phi}{1 + \phi} A \right) &= \frac{\xi}{1 + \phi} A^2 \\ -A^2 \frac{\phi}{1 + \phi} (1 + \xi) + \left[1 + \xi + \frac{\phi}{1 + \phi} \right] A - 1 &= \frac{\xi}{1 + \phi} A^2 \\ \left[\frac{\xi}{1 + \phi} + \frac{\phi}{1 + \phi} (1 + \xi) \right] A^2 - \left[1 + \xi + \frac{\phi}{1 + \phi} \right] A + 1 &= 0 \\ [\xi(1 + \phi) + \phi] \frac{A^2}{1 + \phi} - \left[1 + \xi + \frac{\phi}{1 + \phi} \right] A + 1 &= 0 \\ \left[\xi + \frac{\phi}{1 + \phi} \right] A^2 - \left[1 + \xi + \frac{\phi}{1 + \phi} \right] A + 1 &= 0 \\ \left[\xi + \frac{\phi}{1 + \phi} \right] A(A - 1) - (A - 1) &= 0 \\ \left(\left[\xi + \frac{\phi}{1 + \phi} \right] A - 1 \right) (A - 1) &= 0 \end{aligned}$$

Thus we have two solutions, $A = 1$ and

$$A = \frac{1}{\xi + \frac{\phi}{1 + \phi}} \quad (28)$$

$$B = \frac{-\xi \left(y_0 + \frac{y_1 - \frac{R}{\phi} \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A\phi}{1+\phi} [y_0 + B + \frac{y_1 + \phi B}{R}] - B \right)}{1 + \xi}$$

$$B = -\frac{\xi}{1 + \xi} \left(y_0 + \frac{y_1 - B}{R} \right) + \frac{\xi}{1 + \xi} \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1 + \phi} \left[y_0 + B + \frac{y_1 + \phi B}{R} \right]$$

$$B = \left[\frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1 + \phi} - 1 \right] \frac{\xi}{1 + \xi} \left(y_0 + \frac{y_1}{R} \right) + \frac{\xi}{1 + \xi} \left[\frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1 + \phi} \left(1 + \frac{\phi}{R} \right) + \frac{1}{R} \right] B \quad (29)$$

Note that if $A = 1$,

$$\frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1 + \phi} - 1 = \frac{1}{1 + \phi - \phi} - 1 = 0,$$

so (29) implies $B = 0$. Thus with this solution $\tau(x) = x$, and $c_{t,t} = x_{t,t} - \tau(x_{t,t}) = 0$. Thus we can ignore this solution and assume A is given by (28).

$$B = \frac{\left[\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} - 1 \right] \frac{\xi}{1+\xi}}{1 - \frac{\xi}{1+\xi} \left[\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} \left(1 + \frac{\phi}{R} \right) + \frac{1}{R} \right]} \left(y_0 + \frac{y_1}{R} \right) \quad (30)$$

$$\frac{A}{1+\phi} = \frac{1}{\xi + \phi + \phi\xi}$$

$$1 - \frac{\phi}{1+\phi}A = 1 - \frac{\phi}{\xi + \phi + \phi\xi} = \frac{\xi + \phi\xi}{\xi + \phi + \phi\xi} = \frac{\xi(1+\phi)}{\xi + \phi + \phi\xi}$$

$$\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} = \frac{1}{\xi(1+\phi)}$$

$$\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} - 1 = \frac{1 - \xi - \phi\xi}{\xi(1+\phi)}$$

Thus we have

$$B = \frac{\frac{1 - \xi - \phi\xi}{(1+\phi)(1+\xi)}}{1 - \frac{\xi}{1+\xi} \left[\frac{1}{\xi(1+\phi)} \left(1 + \frac{\phi}{R} \right) + \frac{1}{R} \right]} \left(y_0 + \frac{y_1}{R} \right)$$

$$B = \frac{1 - \xi - \phi\xi}{(1+\phi)(1+\xi) - \xi \left[\frac{1}{\xi} \left(1 + \frac{\phi}{R} \right) + \frac{1+\phi}{R} \right]} \left(y_0 + \frac{y_1}{R} \right)$$

$$B = \frac{1 - \xi - \phi\xi}{(1+\phi)(1+\xi) - \left(1 + \frac{\phi}{R} \right) - \frac{(1+\phi)\xi}{R}} \left(y_0 + \frac{y_1}{R} \right)$$

$$= \frac{1 - \xi - \phi\xi}{(\phi + \xi + \phi\xi)(1 - \frac{1}{R})} \left(y_0 + \frac{y_1}{R} \right)$$

$$B = \frac{R}{r} \frac{1 - \xi - \phi\xi}{\phi + \xi + \phi\xi} \left(y_0 + \frac{y_1}{R} \right) \quad (31)$$

Since we can rewrite (28) as

$$A = \frac{1}{\xi + \frac{\phi}{1+\phi}} = \frac{1+\phi}{\xi + \phi + \xi\phi} \quad (32)$$

we have

$$\frac{B}{A} = \frac{1 - \xi - \phi\xi}{1+\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right)$$

$$\frac{B}{A} = \left(\frac{1}{1+\phi} - \xi \right) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right). \quad (33)$$

Using (21), we have

$$\xi(1+\phi) = \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}$$

can further rewrite this as

$$\frac{B}{A} = \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{1 + \phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \quad (34)$$

and

$$A = \frac{1 + \phi}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \quad (35)$$

$$B = \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \quad (36)$$

Note that (29) simplifies to

$$B = \frac{\xi}{1 + \xi} \left\{ \left[\frac{1}{\xi(1 + \phi)} - 1 \right] \left(y_0 + \frac{y_1}{R}\right) + \left[\frac{1}{\xi(1 + \phi)} \left(1 + \frac{\phi}{R}\right) + \frac{1}{R} \right] B \right\}$$

As a check of (36), substituting it into this equation we get

$$\begin{aligned} B &= \frac{\xi}{1 + \xi} \left[\left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} - 1 \right] \left(y_0 + \frac{y_1}{R}\right) \\ &\quad + \frac{\xi}{1 + \xi} + \left[\left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left(1 + \frac{\phi}{R}\right) + \frac{1}{R} \right] \frac{1 - \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma}} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \end{aligned}$$

$$\begin{aligned} B &= \frac{\left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma} \frac{1}{1 + \phi}}{1 + \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma} \frac{1}{1 + \phi}} \left\{ \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left[\phi + \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma} \right] \frac{R - 1}{R} \right. \\ &\quad \left. + \left[\left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left(1 + \frac{\phi}{R}\right) + \frac{1}{R} \right] \right\} \frac{1 - \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma}} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \end{aligned}$$

$$\begin{aligned} 1 + \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} (1 + \phi) &= \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left[\phi + \left(\frac{\chi_c}{1 + \phi}\right)^{-1/\gamma} \right] \frac{R - 1}{R} \\ &\quad + \left[\left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left(1 + \frac{\phi}{R}\right) + \frac{1}{R} \right] \end{aligned}$$

$$\begin{aligned} 1 + \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} (1 + \phi) &= \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} \left[\phi \left(1 - \frac{1}{R}\right) + 1 + \frac{\phi}{R} \right] + 1 - \frac{1}{R} + \frac{1}{R} \\ &= \left(\frac{\chi_c}{1 + \phi}\right)^{1/\gamma} [1 + \phi] + 1 \end{aligned}$$

$$\begin{aligned}
c_{t,t-1} &= x_{t,t-1} - \tau_t = x_{t,t-1} - (Ax_{t,t-1} + B) = (1-A)x_{t,t-1} - B \\
&= x_{t,t-1} - \frac{1}{\xi + \frac{\phi}{1+\phi}} \left(x_{t,t-1} + \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{1+\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \right) \\
&= x_{t,t-1} - \frac{1+\phi}{\phi + \xi + \xi\phi} \left(x_{t,t-1} + \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{1+\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \right) \\
&= \frac{\xi(1+\phi) - 1}{\phi + \xi + \phi\xi} x_{t,t-1} - \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{\phi + \xi + \xi\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \\
c_{t,t-1} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \left[x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \right] \tag{37}
\end{aligned}$$

This is the sort of behavior we would expect in a Barro (1974) model, where consumption is a multiple of the total wealth of the dynasty. In the limit as $\chi_c \rightarrow 0$ so the parent does not care about the child's consumption,

$$c_{t,t-1} \rightarrow x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right),$$

as it should. The parent consumes the whole wealth of the dynasty, expecting each ensuing generation to pay the debts of the previous generation.

The young agent's consumption is

$$c_{t,t} = \frac{\phi}{1+\phi} \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} + \tau_t \right)$$

From (25),

$$\begin{aligned}
c_{t,t} &= \frac{\phi}{1+\phi} \left(y_0 + \frac{y_1 - \frac{R}{\phi} \frac{1}{1-\frac{\phi}{1+\phi}A} \frac{A\phi}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] - B}{R} + \tau_t \right) \\
c_{t,t} &= \frac{\phi}{1+\phi} \left(y_0 + \frac{y_1 - \frac{R}{\phi} \frac{1}{1-\frac{\phi}{1+\phi}A} \frac{A\phi}{1+\phi} \left[y_0 + Ax_{t,t-1} + B + \frac{y_1 + \phi B}{R} \right] - B}{R} + Ax_{t,t-1} + B \right) \\
c_{t,t} &= \frac{\phi}{1+\phi} \left[\left(1 - \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} \right) \left(y_0 + \frac{y_1}{R} + Ax_{t,t-1} \right) \right. \\
&\quad \left. + \left[1 - \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} \left(1 + \frac{\phi}{R} \right) - \frac{1}{R} \right] B \right]
\end{aligned}$$

From (29),

$$\frac{1+\xi}{\xi} B = \left[\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} - 1 \right] \left(y_0 + \frac{y_1}{R} \right) + \left[\frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} \left(1 + \frac{\phi}{R} \right) + \frac{1}{R} \right] B. \tag{38}$$

Thus

$$\begin{aligned}
c_{t,t} &= \frac{\phi}{1+\phi} \left[\left(1 - \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1+\phi} \right) A x_{t,t-1} + B - \frac{1+\xi}{\xi} B \right] \\
&= \frac{\phi}{1+\phi} \left[\left(1 - \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{A}{1+\phi} \right) A x_{t,t-1} - \frac{B}{\xi} \right] \\
&= \frac{\phi}{1+\phi} \left[\left(1 - \frac{1}{\xi(1+\phi)} \right) \frac{1}{\xi + \frac{\phi}{1+\phi}} x_{t,t-1} - \frac{B}{\xi} \right] \\
&= \frac{\phi}{1+\phi} \left[\left(1 - \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} \right) \frac{1+\phi}{\xi(1+\phi) + \phi} x_{t,t-1} - B(1+\phi) \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} \right] \\
&= \phi \left[\frac{1 - \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma}}{\phi + \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}} x_{t,t-1} - \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} \frac{1 - \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \phi \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} \frac{\left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \tag{39}
\end{aligned}$$

Thus

$$\frac{c_{t,t}}{c_{t,t-1}} = \phi \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} = \frac{\phi}{1+\phi} \frac{1}{\xi} \tag{40}$$

as is required by the intergenerational Euler equation.

$$\begin{aligned}
c_{t+1,t} &= x_{t+1,t} - \tau_{t+1} \\
&= \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] \\
&\quad - A \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] - B \\
&= (1-A) \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left[y_0 + \tau_t + \frac{y_1 + \phi B}{R} \right] - B \\
&= (1-A) \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left[y_0 + A x_{t,t-1} + B + \frac{y_1 + \phi B}{R} \right] - B \\
&= (1-A) \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left[y_0 + A x_{t,t-1} + \frac{y_1}{R} \right] \\
&\quad + \left((1-A) \frac{1}{1 - \frac{\phi}{1+\phi} A} \frac{R}{1+\phi} \left(1 + \frac{\phi}{R} \right) - 1 \right) B \\
A &= \frac{1+\phi}{\phi + \left(\frac{\chi_c}{1+\phi} \right)^{-1/\gamma}}
\end{aligned}$$

$$1 - A = \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}$$

$$1 - \frac{\phi}{1+\phi}A = 1 - \frac{\phi}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} = \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} = \frac{1}{1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}}$$

$$\begin{aligned} c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \left(1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \frac{R}{1+\phi} \left[y_0 + \frac{1+\phi}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} x_{t,t-1} + \frac{y_1}{R} \right] \\ &+ \left(\frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \left(1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \frac{R}{1+\phi} \left(1 + \frac{\phi}{R}\right) - 1 \right) \frac{1 - \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \end{aligned}$$

$$\begin{aligned} c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left\{ \left(1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \left[y_0 + \frac{1+\phi}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} x_{t,t-1} + \frac{y_1}{R} \right] \right. \\ &\left. + \left(\frac{1+\phi}{R} - \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \left(1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \frac{R+\phi}{R} \right) \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \right\} \end{aligned}$$

$$\begin{aligned} c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left\{ \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} (1+\phi)x_{t,t-1} \right. \\ &\left. + \left[1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} + \left(\frac{1+\phi}{R} - \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \left(1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \frac{R+\phi}{R} \right) \frac{R}{r} \right] \left(y_0 + \frac{y_1}{R}\right) \right\} \end{aligned}$$

$$\begin{aligned} c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left\{ \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} (1+\phi)x_{t,t-1} \right. \\ &\left. + \left[1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} + \left(\frac{1+\phi}{R} - \left(\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1 \right) \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} \frac{R-1+1+\phi}{R} \right) \frac{R}{r} \right] \left(y_0 + \frac{y_1}{R}\right) \right\} \end{aligned}$$

$$\begin{aligned} c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left\{ \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} (1+\phi)x_{t,t-1} \right. \\ &\left. + \left[1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} + \left(\frac{1+\phi}{R} - \left(1 - \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} \right) \frac{R-1+1+\phi}{R} \right) \frac{R}{r} \right] \left(y_0 + \frac{y_1}{R}\right) \right\} \end{aligned}$$

$$\begin{aligned}
c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left\{ \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} (1+\phi)x_{t,t-1} \right. \\
&\quad \left. + \left[1 + \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} - \left(1 - \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) + \left(\frac{1+\phi}{R} - \left(1 - \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma}\right) \frac{1+\phi}{R}\right) \frac{R}{r} \right] \left(y_0 + \frac{y_1}{R}\right) \right\} \\
c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} \frac{R}{1+\phi} \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} (1+\phi) \left\{ x_{t,t-1} + \left(1 + \frac{1}{r}\right) \left(y_0 + \frac{y_1}{R}\right) \right\} \\
c_{t+1,t} &= \frac{\left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma} - 1}{\phi + \left(\frac{\chi_c}{1+\phi}\right)^{-1/\gamma}} R \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} \left[x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \right]. \tag{41}
\end{aligned}$$

Thus $\frac{c_{t+1,t}}{c_{t,t}} = \frac{R}{\phi} = \frac{R}{(\beta R^{1-\gamma})^{-1/\gamma}} = \frac{1}{(\beta R)^{-1/\gamma}} = (\beta R)^{1/\gamma}$, so the intertemporal Euler equation is also satisfied. From (40), in a steady state we need

$$\frac{c_{t,t}}{c_{t,t-1}} = \phi \left(\frac{\chi_c}{1+\phi}\right)^{1/\gamma} = 1.$$

For $\gamma = 1$, we need

$$\chi_c = \frac{1+\phi}{\phi} = \frac{1+\beta^{-1}}{\beta^{-1}} = 1+\beta. \tag{42}$$

1.2 Pure Altruism Model

As an alternative to the paterfamilias model of 1.1, let us consider the pure altruism model of Barro (1974). Let ρ be the intergenerational discount factor. A household that receives the transfer τ solves the Bellman equation

$$v(\tau) = \max_{c_0, c_1, \tau'} u(c_0) + \beta u(c_1) + \rho v(\tau') \tag{43}$$

subject to

$$\begin{aligned}
c_0 + b &= y_0 + \tau \\
c_1 + \tau' &= y_1 + Rb.
\end{aligned}$$

Let us guess that

$$v(\tau) = Du(\tau + F) + G. \tag{44}$$

The Lagrangian is

$$L = u(y_0 + \tau - b) + \beta u(y_1 + Rb - \tau') + \rho [Du(\tau' + F) + G].$$

$$\begin{aligned}
\frac{\partial L}{\partial b} &= -(y_0 + \tau - b)^{-\gamma} + \beta R(y_1 + Rb - \tau')^{-\gamma} = 0 \\
\frac{\partial L}{\partial \tau'} &= -\beta(y_1 + Rb - \tau')^{-\gamma} + \rho D(\tau' + F)^{-\gamma} = 0
\end{aligned}$$

$$\begin{aligned}
y_0 + \tau - b &= (\beta R)^{-1/\gamma} (y_1 + Rb - \tau') \\
y_1 + Rb - \tau' &= \left(\frac{\rho D}{\beta} \right)^{-1/\gamma} (\tau' + F) \\
y_0 + \tau - b &= \phi \left(b + \frac{y_1 - \tau'}{R} \right)
\end{aligned}$$

Let us define

$$\zeta = \left(\frac{\rho D}{\beta} \right)^{-1/\gamma} \quad (45)$$

$$y_1 + Rb - \tau' = \zeta (\tau' + F)$$

$$\begin{aligned}
(1 + \phi)b - \frac{\phi}{R}\tau' &= y_0 + \tau - \frac{\phi}{R}y_1 \\
(1 + \zeta)\tau' - Rb &= y_1 - \zeta F
\end{aligned}$$

$$\begin{aligned}
(1 + \phi)Rb - \phi\tau' &= R(y_0 + \tau) - \phi y_1 \\
(1 + \zeta)(1 + \phi)\tau' - R(1 + \phi)b &= (1 + \phi)y_1 - \zeta(1 + \phi)F
\end{aligned}$$

$$[1 + \zeta + \zeta\phi]\tau' = (1 + \phi)y_1 - \zeta(1 + \phi)F + R(y_0 + \tau) - \phi y_1$$

$$[1 + \zeta + \zeta\phi]\tau' = R(y_0 + \tau) + y_1 - \zeta(1 + \phi)F$$

$$\tau' = \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 - \zeta(1 + \phi)F}{R} \right] \quad (46)$$

$$b = \frac{1}{1 + \phi} \left[y_0 + \tau - \frac{\phi}{R} (y_1 - \tau') \right]$$

$$b = \frac{1}{1 + \phi} \left[y_0 + \tau - \frac{\phi}{R} \left(y_1 - \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 - \zeta(1 + \phi)F}{R} \right] \right) \right]$$

$$b = \frac{1}{1 + \phi} \left[y_0 + \tau - \frac{\phi}{1 + \zeta(1 + \phi)} \left[-y_0 - \tau + \frac{\zeta(1 + \phi)(y_1 + F)}{R} \right] \right]$$

$$b = \frac{1}{1 + \phi} \frac{1}{1 + \zeta(1 + \phi)} \left[(1 + \phi)(1 + \zeta)(y_0 + \tau) - \frac{\phi\zeta(1 + \phi)(y_1 + F)}{R} \right]$$

$$b = \frac{1}{1 + \zeta(1 + \phi)} \left[(1 + \zeta)(y_0 + \tau) - \zeta \frac{\phi}{R} (y_1 + F) \right] \quad (47)$$

$$c_0 = y_0 + \tau - b = y_0 + \tau - \frac{1}{1 + \zeta(1 + \phi)} \left[(1 + \zeta)(y_0 + \tau) - \zeta \frac{\phi}{R} (y_1 + F) \right]$$

$$= \frac{1}{1 + \zeta(1 + \phi)} \left[[1 + \zeta + \zeta\phi - (1 + \zeta)] (y_0 + \tau) + \zeta \frac{\phi}{R} (y_1 + F) \right]$$

$$c_0 = \frac{\phi\zeta}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 + F}{R} \right] \quad (48)$$

$$\begin{aligned}
c_1 &= y_1 + Rb - \tau' \\
&= y_1 + \frac{R}{1 + \zeta(1 + \phi)} \left[(1 + \zeta)(y_0 + \tau) - \zeta \frac{\phi}{R}(y_1 + F) \right] \\
&\quad - \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 - \zeta(1 + \phi)F}{R} \right] \\
c_1 &= \frac{R}{1 + \zeta(1 + \phi)} \left[\frac{1 + \zeta + \zeta\phi}{R} y_1 + (1 + \zeta)(y_0 + \tau) - \zeta \frac{\phi}{R}(y_1 + F) - y_0 - \tau - \frac{y_1 - \zeta(1 + \phi)F}{R} \right] \\
c_1 &= \frac{R}{1 + \zeta(1 + \phi)} \left[\zeta(y_0 + \tau) + \frac{(1 + \zeta + \zeta\phi)y_1 - \zeta\phi(y_1 + F) - y_1 + \zeta(1 + \phi)F}{R} \right] \\
c_1 &= \frac{R}{1 + \zeta(1 + \phi)} \left[\zeta(y_0 + \tau) + \frac{\zeta y_1 + \zeta F}{R} \right] \\
c_1 &= \frac{\zeta R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 + F}{R} \right] \tag{49}
\end{aligned}$$

$$\begin{aligned}
\tau' + F &= \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 - \zeta(1 + \phi)F}{R} \right] + F \\
&= \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 - \zeta(1 + \phi)F}{R} + \frac{1 + \zeta(1 + \phi)F}{R} \right] \\
\tau' + F &= \frac{R}{1 + \zeta(1 + \phi)} \left[y_0 + \tau + \frac{y_1 + F}{R} \right] \tag{50}
\end{aligned}$$

For the ansatz to be correct, we need

$$\begin{aligned}
\tau + F &= y_0 + \tau + \frac{y_1 + F}{R} \\
\left(1 - \frac{1}{R}\right) F &= y_0 + \frac{y_1}{R} \\
F &= \frac{R}{r} \left(y_0 + \frac{y_1}{R}\right) \tag{51}
\end{aligned}$$

For $\gamma \neq 1$, the Bellman equation becomes

$$\begin{aligned}
D \frac{(\tau + F)^{1-\gamma}}{1-\gamma} + G &= \frac{1}{1-\gamma} \left(\frac{\phi\zeta}{1 + \zeta(1 + \phi)} [\tau + F] \right)^{1-\gamma} + \frac{\beta}{1-\gamma} \left(\frac{\zeta R}{1 + \zeta(1 + \phi)} [\tau + F] \right)^{1-\gamma} \\
&\quad + \frac{\rho D}{1-\gamma} \left(\frac{R}{1 + \zeta(1 + \phi)} [\tau + F] \right)^{1-\gamma} + \rho G
\end{aligned}$$

For $\rho \neq 1$, we must have $G = 0$.

$$\begin{aligned}
D &= \left(\frac{\phi\zeta}{1 + \zeta(1 + \phi)} \right)^{1-\gamma} + \beta \left(\frac{\zeta R}{1 + \zeta(1 + \phi)} \right)^{1-\gamma} + \rho D \left(\frac{R}{1 + \zeta(1 + \phi)} \right)^{1-\gamma} \\
D &= \frac{(\phi\zeta)^{1-\gamma} + \beta(\zeta R)^{1-\gamma} + \rho DR^{1-\gamma}}{(1 + \zeta + \zeta\phi)^{1-\gamma}}
\end{aligned}$$

From (45),

$$\zeta = \left(\frac{\rho D}{\beta} \right)^{-1/\gamma}$$

$$D = \frac{\beta}{\rho} \zeta^{-\gamma}$$

$$D = \frac{(\phi\zeta)^{1-\gamma} + \zeta^{1-\gamma}\phi^{-\gamma} + \frac{\rho D}{\beta}\beta R^{1-\gamma}}{(1 + \zeta + \zeta\phi)^{1-\gamma}}$$

$$= \frac{(\phi\zeta)^{1-\gamma} + \zeta^{1-\gamma}\phi^{-\gamma} + \zeta^{-\gamma}\phi^{-\gamma}}{(1 + \zeta + \zeta\phi)^{1-\gamma}}$$

$$= (\phi\zeta)^{-\gamma} \frac{\phi\zeta + \zeta + 1}{(1 + \zeta + \zeta\phi)^{1-\gamma}}$$

$$D = \left(\frac{\phi\zeta}{1 + \zeta + \phi\zeta} \right)^{-\gamma}$$

$$D^{-1/\gamma} = \frac{\phi\zeta}{1 + \zeta + \phi\zeta}$$

$$\zeta = \left(\frac{\rho}{\beta} \right)^{-1/\gamma} \frac{\phi\zeta}{1 + \zeta + \phi\zeta}$$

$$1 + (1 + \phi)\zeta = \left(\frac{\rho}{\beta} \right)^{-1/\gamma} \phi = \left(\frac{\rho}{\beta} \beta R^{1-\gamma} \right)^{-1/\gamma} = (\rho R^{1-\gamma})^{-1/\gamma} \quad (52)$$

$$\zeta = \frac{\left(\frac{\rho}{\beta} \beta R^{1-\gamma} \right)^{-1/\gamma} - 1}{1 + \phi}$$

$$\zeta = \frac{(\rho R^{1-\gamma})^{-1/\gamma} - 1}{1 + \phi} \quad (53)$$

We need

$$(\rho R^{1-\gamma})^{-1/\gamma} > 1$$

in order for ζ to be positive, so we need

$$\rho R^{1-\gamma} < 1$$

That is, we need

$$\rho < R^{\gamma-1} \quad (54)$$

in order for the pure altruism model to have a well-defined solution.

$$D = \frac{\beta}{\rho} \left(\frac{(\rho R^{1-\gamma})^{-1/\gamma} - 1}{1 + \phi} \right)^{-\gamma} = \frac{\beta}{\rho} \left(\frac{1 + \phi}{(\rho R^{1-\gamma})^{-1/\gamma} - 1} \right)^{\gamma}. \quad (55)$$

Let us define

$$\psi = (\rho R^{1-\gamma})^{-1/\gamma} = 1 + \zeta + \phi\zeta, \quad (56)$$

which is the intergenerational analog of ϕ . Then we have

$$\begin{aligned} c_0 &= \frac{\phi\zeta}{\psi} [\tau + F] \\ c_1 &= \frac{R\zeta}{\psi} [\tau + F], \end{aligned}$$

so

$$\frac{c_1}{c_0} = \frac{R}{\phi} = (\beta R)^{1/\gamma},$$

and the intertemporal Euler equation is satisfied.

$$\frac{c'_0}{c_0} = \frac{\frac{\phi\zeta}{\psi}(\tau' + F)}{\frac{\phi\zeta}{\psi}(\tau + F)} = \frac{R}{1 + \zeta(1 + \phi)} = \frac{R}{\psi} = (\rho R)^{1/\gamma},$$

so the intergenerational Euler equation is satisfied.

In a steady-state of the pure altruism model, we have to have $\rho = \frac{1}{R}$. This is problematic because for a thirty-year period, $\rho = \frac{1}{1.04^{30}} = .31$. It is hard to see how people could evolve if they put so little weight on the utility of their children. This is in contrast to the paterfamilias model, where a steady state requires people to put more weight on the consumption of their children than themselves.

The mapping between the two models is

$$\begin{aligned} (\rho R)^{1/\gamma} &= \phi \left(\frac{\chi_c}{1 + \phi} \right)^{1/\gamma} = \left(\frac{\chi_c}{(1 + \phi)\beta R^{1-\gamma}} \right)^{1/\gamma} \\ \rho R &= \frac{\chi_c}{(1 + \phi)\beta R^{1-\gamma}} \\ \chi_c &= \rho\beta(1 + \phi)R^{2-\gamma}. \end{aligned} \tag{57}$$

Since we have to have

$$\rho R^{1-\gamma} < 1,$$

only paterfamilias models with

$$\chi_c < \beta R(1 + \phi)$$

can map into the pure altruism model. Though some have characterised the paterfamilias model as a reduced-form version of the pure altruism model, in fact the pure altruism model is a special case of the paterfamilias model.

Indeed, the two models ought to be easily distinguished experimentally. If you isolate a parent and child together and offer them a candy bar, how will it be divided between them? The pure altruism model predicts the parent will give the kid about 25% of the candy bar. The paterfamilias model predicts the parent will give the kid more than half of the candy bar.

1.3 Logan's Run Model

Next we consider the Logan's Run model where children have all the bargaining power. Then we have

$$A_{21} = \chi_p \tag{58}$$

$$A_{22} = (1 + \phi^{-1})^\gamma \tag{59}$$

We still have

$$\begin{aligned} D_1 &= 1 \\ D_2 &= \frac{\phi}{1+\phi}. \end{aligned}$$

Then we have

$$\tau_t = \frac{\chi_p^{-1/\gamma} x_{t,t-1} - \frac{\phi}{1+\phi} \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right)}{\chi_p^{-1/\gamma} + \frac{\phi}{1+\phi}}$$

Let us define

$$\xi_p = \chi_p^{-1/\gamma} \quad (60)$$

$$\tau_t = \frac{(1+\phi)\xi_p x_{t,t-1} - \phi \left(y_0 + \frac{y_1 - \tau_{t+1}}{R} \right)}{\phi + (1+\phi)\xi_p} \quad (61)$$

$$\begin{aligned} c_{t,t-1} &= \frac{\frac{\phi}{1+\phi}}{\chi_p^{-1/\gamma} + \frac{\phi}{1+\phi}} \left(x_{t,t-1} + y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \\ c_{t,t-1} &= \frac{\phi}{\phi + (1+\phi)\xi_p} \left(x_{t,t-1} + y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \end{aligned} \quad (62)$$

$$\begin{aligned} c_{t,t} &= \frac{\phi}{1+\phi} \frac{\chi_p^{-1/\gamma}}{\chi_p^{-1/\gamma} + \frac{\phi}{1+\phi}} \left(x_{t,t-1} + y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \\ c_{t,t} &= \frac{\phi\xi_p}{\phi + (1+\phi)\xi_p} \left(x_{t,t-1} + y_0 + \frac{y_1 - \tau_{t+1}}{R} \right) \end{aligned} \quad (63)$$

Note that

$$\frac{c_{t,t}}{c_{t,t-1}} = \xi_p. \quad (64)$$

Let us suppose again that (23) still holds. Then (24) and (25) still follow since they only depend on (23). If we substitute (25) into (61), we get

$$Ax_{t,t-1} + B = \frac{(1+\phi)\xi_p x_{t,t-1} - \phi \left(y_0 + \frac{y_1}{R} - \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{A}{1+\phi} \left[y_0 + Ax_{t,t-1} + B + \frac{y_1 + \phi B}{R} \right] - \frac{B}{R} \right)}{\phi + (1+\phi)\xi_p}$$

$$A = \frac{(1+\phi)\xi_p + \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{\phi}{1+\phi} A^2}{\phi + (1+\phi)\xi_p}$$

$$(1+\phi)\xi_p(A-1) + \phi A = \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{\phi}{1+\phi} A^2 = \frac{\phi A^2}{1 + \phi(1-A)}$$

$$\begin{aligned} [(1+\phi)\xi_p(A-1) + \phi A][1 + \phi(1-A)] &= \phi A^2 \\ (1+\phi)\xi_p(A-1)[1 + \phi(1-A)] + \phi A + \phi^2 A(1-A) &= \phi A^2 \\ (1+\phi)\xi_p(A-1)[1 + \phi(1-A)] + \phi A(1-A) + \phi^2 A(1-A) &= 0 \\ (1+\phi)[\xi_p(A-1)[1 + \phi(1-A)] + \phi A(1-A)] &= 0 \end{aligned}$$

$$(A-1)[\xi_p[1+\phi(1-A)]-\phi A]=0$$

$$B = \frac{-\phi \left(y_0 + \frac{y_1}{R} - \frac{1}{1-\frac{\phi}{1+\phi}A} \frac{1}{1+\phi} \left[y_0 + B + \frac{y_1+\phi B}{R} \right] - \frac{B}{R} \right)}{\phi + (1+\phi)\xi_p} \quad (65)$$

If $A = 1$, (65) reduces to

$$B = \frac{-\phi \left(y_0 + \frac{y_1}{R} - \frac{1}{1-\frac{\phi}{1+\phi}} \frac{1}{1+\phi} \left[y_0 + B + \frac{y_1+\phi B}{R} \right] - \frac{B}{R} \right)}{\phi + (1+\phi)\xi_p}$$

$$= \frac{-\phi \left(y_0 + \frac{y_1}{R} - \left[y_0 + B + \frac{y_1+\phi B}{R} \right] - \frac{B}{R} \right)}{\phi + (1+\phi)\xi_p}$$

$$= \frac{-\phi \left(- \left[1 + \frac{\phi}{R} \right] B - \frac{B}{R} \right)}{\phi + (1+\phi)\xi_p},$$

which implies $B = 0$. Since this would imply $\tau_t = x_{t,t-1}$, we can again rule out this solution.

$$\xi_p[1+\phi(1-A)]-\phi A = 0$$

$$\xi_p(1+\phi) - (\xi_p+1)\phi A = 0$$

$$A = \frac{\xi_p}{\xi_p+1} \frac{1+\phi}{\phi} \quad (66)$$

$$\frac{1}{1-\frac{\phi}{1+\phi}A} \frac{1}{1+\phi} = \frac{1}{1-\frac{\xi_p}{\xi_p+1}} \frac{1}{\phi} \frac{\xi_p}{1+\xi_p} = \frac{1}{\frac{1}{1+\xi_p}} \frac{1}{\phi} \frac{\xi_p}{1+\xi_p} = \frac{\xi_p}{\phi}.$$

$$[\phi + (1+\phi)\xi_p]B = -\phi \left(y_0 + \frac{y_1}{R} - \frac{\xi_p}{\phi} \left[y_0 + B + \frac{y_1+\phi B}{R} \right] - \frac{B}{R} \right)$$

$$= (\xi_p - \phi) \left(y_0 + \frac{y_1}{R} \right) + \left[\xi_p + (1+\xi_p) \frac{\phi}{R} \right] B$$

$$\left[\phi + (1+\phi)\xi_p - \xi_p - (1+\xi_p) \frac{\phi}{R} \right] B = (\xi_p - \phi) \left(y_0 + \frac{y_1}{R} \right)$$

$$\left[\phi(1+\xi_p) - (1+\xi_p) \frac{\phi}{R} \right] B = (\xi_p - \phi) \left(y_0 + \frac{y_1}{R} \right)$$

$$(1+\xi_p)\phi \frac{r}{R} B = (\xi_p - \phi) \left(y_0 + \frac{y_1}{R} \right)$$

$$B = \frac{R}{r} \frac{\xi_p - \phi}{(1+\xi_p)\phi} \left(y_0 + \frac{y_1}{R} \right) \quad (67)$$

$$\frac{B}{A} = \frac{\xi_p+1}{\xi_p} \frac{\phi}{1+\phi} \frac{R}{r} \frac{\xi_p - \phi}{(1+\xi_p)\phi} \left(y_0 + \frac{y_1}{R} \right)$$

$$\frac{B}{A} = \frac{\xi_p - \phi}{(1+\phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \quad (68)$$

Thus

$$\tau_t = \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right]. \quad (69)$$

The x map becomes

$$\begin{aligned} x_{t+1,t} &= \frac{1}{1 - \frac{\phi}{1+\phi}A} \frac{R}{1 + \phi} \left[Ax_{t,t-1} + y_0 + \frac{y_1}{R} + \left(1 + \frac{\phi}{R}\right) B \right] \\ &= \frac{1}{1 - \frac{\xi_p}{1+\xi_p}} \frac{R}{1 + \phi} \left[\frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} x_{t,t-1} + y_0 + \frac{y_1}{R} + \left(1 + \frac{\phi}{R}\right) \frac{R}{r} \frac{\xi_p - \phi}{(1 + \xi_p)\phi} \left(y_0 + \frac{y_1}{R} \right) \right] \\ &= R \frac{1 + \xi_p}{1 + \phi} \left[\frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} x_{t,t-1} + \left[1 + \left(1 + \frac{\phi}{R}\right) \frac{R}{r} \frac{\xi_p - \phi}{(1 + \xi_p)\phi} \right] \left(y_0 + \frac{y_1}{R} \right) \right] \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + R \frac{1 + \xi_p}{1 + \phi} \left[1 + \left(1 + \frac{\phi}{R}\right) \frac{R}{r} \frac{\xi_p - \phi}{(1 + \xi_p)\phi} \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + \frac{R}{1 + \phi} \left[1 + \xi_p + \frac{R + \phi}{R - 1} \left(\frac{\xi_p}{\phi} - 1 \right) \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + \frac{R}{1 + \phi} \left[1 + \xi_p + \frac{\xi_p}{\phi} - 1 + \frac{1 + \phi}{R - 1} \left(\frac{\xi_p}{\phi} - 1 \right) \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + \frac{R}{1 + \phi} \left[\frac{1 + \phi}{\phi} \xi_p + \frac{1 + \phi}{R - 1} \left(\frac{\xi_p}{\phi} - 1 \right) \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + R \left[\frac{1}{\phi} \xi_p + \frac{1}{R - 1} \left(\frac{\xi_p}{\phi} - 1 \right) \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} x_{t,t-1} + \frac{R}{r} \left[\frac{R}{\phi} \xi_p - 1 \right] \left(y_0 + \frac{y_1}{R} \right) \\ x_{t+1,t} &= \xi_p \frac{R}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) - \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \end{aligned} \quad (70)$$

$$\begin{aligned} \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[\xi_p \frac{R}{\phi} x_{t,t-1} + \frac{R}{r} \left[\frac{R}{\phi} \xi_p - 1 \right] \left(y_0 + \frac{y_1}{R} \right) + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\ \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[\xi_p \frac{R}{\phi} x_{t,t-1} + \left[\frac{R}{\phi} \xi_p - 1 + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \right] \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\ \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[\xi_p \frac{R}{\phi} x_{t,t-1} + \left[\frac{R}{\phi} \xi_p - 1 + \frac{1}{1 + \phi} - \frac{\phi}{(1 + \phi)\xi_p} \right] \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\ \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[\xi_p \frac{R}{\phi} x_{t,t-1} + \left[\frac{R}{\phi} \xi_p - \frac{\phi}{1 + \phi} - \frac{\phi}{(1 + \phi)\xi_p} \right] \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\ \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[\xi_p \frac{R}{\phi} x_{t,t-1} + \left[\frac{R}{\phi} \xi_p - \frac{\phi}{1 + \phi} \frac{1 + \xi_p}{\xi_p} \right] \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\ \tau_{t+1} &= \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \xi_p \frac{R}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) - \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \end{aligned} \quad (71)$$

$$\begin{aligned}
c_{t,t-1} &= x_{t,t-1} - \tau_t = x_{t,t-1} - \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
&= \frac{1}{(\xi_p + 1)\phi} \left[(\xi_p + 1)\phi x_{t,t-1} - \xi_p(1 + \phi)x_{t,t-1} - (\xi_p - \phi) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
&= \frac{1}{(\xi_p + 1)\phi} \left[(\phi - \xi_p)x_{t,t-1} - (\xi_p - \phi) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t-1} &= \frac{\phi - \xi_p}{(\xi_p + 1)\phi} \left[x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right], \tag{72}
\end{aligned}$$

which is again consistent with Barro's (1974) result.

$$\begin{aligned}
c_{t,t} &= \frac{\phi}{1 + \phi} \left[y_0 + \tau_t + \frac{y_1 - \tau_{t+1}}{R} \right] \\
c_{t,t} &= \frac{\phi}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \left[x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \right. \\
&\quad \left. - \frac{\xi_p}{\xi_p + 1} \frac{1 + \phi}{\phi} \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) + \frac{1}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\phi}{1 + \phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) + \frac{\xi_p}{\xi_p + 1} \left[x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) - \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \right] \\
c_{t,t} &= \frac{\phi}{1 + \phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) + \frac{\xi_p}{\xi_p + 1} \left[\frac{\phi - \xi_p}{\phi} x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) - \frac{\xi_p}{\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\xi_p}{\xi_p + 1} \left[\frac{\phi - \xi_p}{\phi} x_{t,t-1} + \frac{\xi_p - \phi}{(1 + \phi)\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) - \frac{\xi_p}{\phi} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) + \frac{\phi}{1 + \phi} \frac{\xi_p + 1}{\xi_p} \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\xi_p}{\xi_p + 1} \left[\frac{\phi - \xi_p}{\phi} x_{t,t-1} + \left(\frac{\xi_p - \phi}{(1 + \phi)\xi_p} - \frac{\xi_p}{\phi} + \frac{\phi}{1 + \phi} \frac{\xi_p + 1}{\xi_p} \right) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\xi_p}{\xi_p + 1} \left[\frac{\phi - \xi_p}{\phi} x_{t,t-1} + \left(-\frac{\xi_p}{\phi} + \frac{\phi\xi_p + \phi + \xi_p - \phi}{\xi_p(1 + \phi)} \right) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\xi_p}{\xi_p + 1} \left[\frac{\phi - \xi_p}{\phi} x_{t,t-1} + \left(-\frac{\xi_p}{\phi} + 1 \right) \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \\
c_{t,t} &= \frac{\xi_p}{\xi_p + 1} \frac{\phi - \xi_p}{\phi} \left[x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right] \tag{73}
\end{aligned}$$

Thus the intergenerational Euler equation

$$\frac{c_{t,t}}{c_{t,t-1}} = \xi_p$$

is satisfied.

$$\begin{aligned}
c_{t+1,t} &= x_{t+1,t} - \tau_{t+1} = \xi_p \frac{R}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) - \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \\
&\quad - \frac{1 + \phi}{\phi} \frac{\xi_p}{1 + \xi_p} \frac{R}{\phi} \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right)
\end{aligned}$$

$$\begin{aligned}
c_{t+1,t} &= \xi_p \frac{R}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) - \frac{1+\phi}{\phi} \frac{\xi_p}{1+\xi_p} R \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \\
c_{t+1,t} &= \left(1 - \frac{1+\phi}{\phi} \frac{\xi_p}{1+\xi_p} \right) R \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \\
c_{t+1,t} &= \left(\frac{\phi + \phi \xi_p - \xi_p - \phi \xi_p}{\phi(1+\xi_p)} \right) R \frac{\xi_p}{\phi} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \\
c_{t+1,t} &= R \frac{\xi_p}{\phi} \frac{\phi - \xi_p}{\phi(1+\xi_p)} \left(x_{t,t-1} + \frac{R}{r} \left(y_0 + \frac{y_1}{R} \right) \right) \tag{74}
\end{aligned}$$

Thus the intertemporal Euler equation

$$\frac{c_{t+1,t}}{c_{t,t}} = \frac{R}{\phi}$$

is also satisfied.

The Logan's Run model is equivalent to the paterfamilias model when

$$\begin{aligned}
\xi_p &= \chi_p^{-1/\gamma} = \phi \left(\frac{\chi_c}{1+\phi} \right)^{1/\gamma} \\
\frac{1}{\chi_p} &= \frac{\chi_c \phi^\gamma}{1+\phi} \\
\chi_p &= \frac{(1 + (\beta R^{1-\gamma})^{-1/\gamma}) \beta R^{1-\gamma}}{\chi_c}.
\end{aligned}$$

So if

$$\chi_p \chi_c = \frac{1+\phi}{\phi^\gamma}, \tag{75}$$

the Logan's Run and paterfamilias models are observationally equivalent. This is not surprising since

$$\begin{aligned}
\frac{A_{12}}{A_{11}} &= \left(\frac{1}{1+\phi} \right)^{1-\gamma} \chi_c \\
\frac{A_{22}}{A_{21}} &= \frac{(1+\phi^{-1})^\gamma}{\chi_p}.
\end{aligned}$$

The condition that the utility weights are equal is

$$\begin{aligned}
\left(\frac{1}{1+\phi} \right)^{1-\gamma} \chi_c &= \left(\frac{1+\phi}{\phi} \right)^\gamma \frac{1}{\chi_p} \\
\chi_c \chi_p &= \left(\frac{1+\phi}{\phi} \right)^\gamma (1+\phi)^{1-\gamma} = \frac{1+\phi}{\phi^\gamma}
\end{aligned}$$

2 Equilibrium with Nash Bargaining

Now we consider what happens when $\theta \in (0, 1)$ so both parties have some say in the value of the transfer. Bargaining is only possible when both parties have something to gain from a transfer so $U_1, U_2 > 0$. We know that $U_1 = U_2 = 0$ when $\tau = 0$. If we plot the utility-possibilities frontier $U_2(U_1)$, we need this to pass through the first quadrant. Thus for small τ , we should have

$$U_2 = mU_1.$$

If $m \leq 0$, we must have $\tau = 0$.

$$m = \lim_{\tau \rightarrow 0} \frac{U_2(\tau)}{U_1(\tau)} = \lim_{\tau \rightarrow 0} \frac{U_2'(\tau)}{U_1'(\tau)} \quad (76)$$

by l'Hôpital's rule.

$$\begin{aligned} U_1'(\tau) &= -A_{11}u'(x_1 - \tau) + A_{12}u'(x_2 + \tau) \\ U_2'(\tau) &= -A_{21}u'(x_1 - \tau) + A_{22}u'(x_2 + \tau) \\ m &= \frac{A_{22}x_2^{-\gamma} - A_{21}x_1^{-\gamma}}{A_{12}x_2^{-\gamma} - A_{11}x_1^{-\gamma}} = \frac{A_{22} - A_{21} \left(\frac{x_2}{x_1}\right)^\gamma}{A_{12} - A_{11} \left(\frac{x_2}{x_1}\right)^\gamma} \end{aligned}$$

Bargaining requires $A_{22} - A_{21} \left(\frac{x_2}{x_1}\right)^\gamma$ and $A_{12} - A_{11} \left(\frac{x_2}{x_1}\right)^\gamma$ to have the same sign.

$$\frac{x_2}{x_1} < \min \left\{ \left(\frac{A_{12}}{A_{11}}\right)^{1/\gamma}, \left(\frac{A_{22}}{A_{21}}\right)^{1/\gamma} \right\}$$

or

$$\frac{x_2}{x_1} > \max \left\{ \left(\frac{A_{12}}{A_{11}}\right)^{1/\gamma}, \left(\frac{A_{22}}{A_{21}}\right)^{1/\gamma} \right\}$$

Let us define

$$\eta_1 = \frac{A_{12}}{A_{11}} \quad (77)$$

and

$$\eta_2 = \frac{A_{22}}{A_{21}} \quad (78)$$

Then there are gains from bargaining if

$$\frac{x_2}{x_1} < \min\{\eta_1^{1/\gamma}, \eta_2^{1/\gamma}\}, \quad (79)$$

in which case the parent will make a transfer to the child with $\tau > 0$, or if

$$\frac{x_2}{x_1} > \max\{\eta_1^{1/\gamma}, \eta_2^{1/\gamma}\}, \quad (80)$$

in which case the child will make a transfer to the parent with $\tau < 0$. When $\eta_1 = \eta_2$, as we saw above, the two sides will have their incentives perfectly aligned and will agree to what is optimal for both of them. The bigger the difference between η_1 and η_2 the less likely it is that they will be able to agree to a transfer. Since in the underlying model,

$$\eta_1 = \left(\frac{1}{1 + \phi}\right)^{1-\gamma} \chi_c$$

and

$$\eta_2 = \left(\frac{\phi}{1 + \phi} \right)^{-\gamma} \frac{1}{\chi_p},$$

we need the child's weight on the parent to be roughly reciprocal to the parent's weight on the child. Thus if the parent and child reciprocate their feelings toward each other, bargaining will be difficult and transfers will rarely occur.

Note that (12) and (13) imply that

$$\begin{aligned} U_1(\tau) &= A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2 + \tau) - u(x_2)] \\ U_2(\tau) &= A_{21}[u(x_1 - \tau) - u(x_1)] + A_{22}[u(x_2 + \tau) - u(x_2)]. \end{aligned}$$

$$\begin{aligned} U_1'(\tau) &= -A_{11}u'(x_1 - \tau) + A_{12}u'(x_2 + \tau) \\ U_2'(\tau) &= -A_{21}u'(x_1 - \tau) + A_{22}u'(x_2 + \tau) \end{aligned}$$

$$\begin{aligned} U_1''(\tau) &= A_{11}u''(x_1 - \tau) + A_{12}u''(x_2 + \tau) < 0 \\ U_2''(\tau) &= A_{21}u''(x_1 - \tau) + A_{22}u''(x_2 + \tau) < 0, \end{aligned}$$

so both $U_1(\tau)$ and $U_2(\tau)$ are strictly concave functions of τ . Suppose it is the case that $U_1'(0)U_2'(0) < 0$. Then there cannot be τ such that $U_1(\tau) > 0$ and $U_2(\tau) > 0$. Suppose that $U_1'(0) < 0 < U_2'(0)$. Since U_1 is strictly concave, $U_1'(\tau) < U_1'(0) < 0$ for $\tau > 0$. Thus U_1 will be strictly decreasing for $\tau > 0$, so $U_1(\tau) < U_1(0) = 0$ for $\tau > 0$. Meanwhile since U_2 is strictly concave, $U_2'(\tau) > U_2'(0) > 0$ for $\tau < 0$. Thus $U_2(\tau)$ is strictly increasing for $\tau < 0$. Then $U_2(\tau) < U_2(0) = 0$ for $\tau < 0$. Thus $U_1(\tau)$ is only positive for $\tau < 0$ and $U_2(\tau)$ is only positive for $\tau > 0$. So there is no τ for which $U_1(\tau) > 0$ and $U_2(\tau) > 0$.

So the condition for there to be a nonzero bargaining solution is, indeed, that

$$\frac{dU_2}{dU_1}(0) = \frac{U_2'(0)}{U_1'(0)} > 0. \quad (81)$$

Let τ_1^* satisfy $U_1'(\tau_1^*) = 0$ and τ_2^* satisfy $U_2'(\tau_2^*) = 0$. Thus

$$\begin{aligned} (x_1 - \tau_1^*)^{-\gamma} &= \eta_1(x_2 + \tau_1^*)^{-\gamma} \\ x_1 - \tau_1^* &= \eta_1^{-1/\gamma}(x_2 + \tau_1^*) \\ \tau_1^* &= \frac{x_1 - \eta_1^{-1/\gamma}x_2}{1 + \eta_1^{-1/\gamma}} \end{aligned} \quad (82)$$

Likewise,

$$\tau_2^* = \frac{x_1 - \eta_2^{-1/\gamma}x_2}{1 + \eta_2^{-1/\gamma}} \quad (83)$$

If (81) is satisfied, τ_1^* and τ_2^* must have the same sign. The optimal τ must be between τ_1^* and τ_2^* .

$$U(\tau) = \theta \ln U_1(\tau) + (1 - \theta) \ln U_2(\tau).$$

$$U'(\tau) = \theta \frac{U_1'(\tau)}{U_1(\tau)} + (1 - \theta) \frac{U_2'(\tau)}{U_2(\tau)}$$

Without loss of generality, we can suppose $\tau_1^* < \tau_2^*$. Then $U_1'(\tau_1^*) = 0 < U_2'(\tau_2^*)$. For $\tau < \tau_1^*$, $U_1'(\tau) > 0$ and $U_2'(\tau) > 0$, so $U(\tau) < U(\tau_1^*)$. Likewise $U_2'(\tau_2^*) = 0 > U_1'(\tau_2^*)$. For $\tau > \tau_2^*$, $U_1'(\tau) < 0$ and $U_2'(\tau) < 0$, so $U'(\tau) < 0$. Therefore $U(\tau) < U(\tau_2^*)$. Note, however that if $0 < \tau_1^* < \tau_2^*$, $U_1(\tau_2^*) \leq 0$ is possible. Likewise if $\tau_1^* < \tau_2^* < 0$, $U_2(\tau_1^*) \leq 0$ is possible. In these cases, we can further constrain the search for τ by looking in the regime where $U_1(\tau) = 0$ or $U_2(\tau) = 0$. If $U_1(\tau) = 0$, then we have

$$u(x_1 - \tau) - u(x_1) + \eta_1[u(x_2 + \tau) - u(x_2)] = 0$$

For $\gamma = 1$, this is

$$\begin{aligned} \ln\left(\frac{x_1 - \tau}{x_1}\right) &= \eta_1 \ln\left(\frac{x_2}{x_2 + \tau}\right) \\ \frac{x_1 - \tau}{x_1} &= \left(\frac{x_2}{x_2 + \tau}\right)^{\eta_1}, \end{aligned}$$

which cannot be solved analytically.

Another way of characterizing (81) is that $0 \notin (\tau_1^*, \tau_2^*)$. Suppose that

$$\tau_1^* < 0 < \tau_2^*.$$

Then

$$\begin{aligned} x_1 - \eta_1^{-1/\gamma} x_2 &< 0 < x_1 - \eta_2^{-1/\gamma} x_2 \\ \eta_1^{1/\gamma} &< \frac{x_2}{x_1} < \eta_2^{1/\gamma}. \end{aligned}$$

Likewise if

$$\begin{aligned} \tau_2^* &< 0 < \tau_1^*, \\ x_1 - \eta_2^{-1/\gamma} x_2 &< 0 < x_1 - \eta_1^{-1/\gamma} x_2, \end{aligned}$$

so

$$\eta_2^{1/\gamma} < \frac{x_2}{x_1} < \eta_1^{1/\gamma}.$$

Eq. (16) simplifies to

$$\begin{aligned} &([u(x_1 - \tau) - u(x_1)] + (\theta\eta_2 + (1 - \theta)\eta_1)[u(x_2 + \tau) - u(x_2)])(x_1 - \tau)^{-\gamma} \\ &= (((1 - \theta)\eta_2 + \theta\eta_1)[u(x_1 - \tau) - u(x_1)] + \eta_1\eta_2[u(x_2 + \tau) - u(x_2)])(x_2 + \tau)^{-\gamma}. \end{aligned} \quad (84)$$

At a given t , the budget constraints are

$$\begin{aligned} c_{t,t}^i + b_{t+1}^i &= y_{t,t}^i + \tau_t^i \\ c_{t,t-1}^i + \tau_t^i &= x_{t,t-1}^i. \end{aligned}$$

We must have $c_{t,t}^i, c_{t,t-1}^i \geq 0$. In partial equilibrium with fixed R , let us assume we impose the no-Ponzi condition

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{b_{t+s}^i}{R^s} &\geq 0. \\ c_{t,t} + b_{t+1}^i + c_{t,t-1}^i &= y_{t,t}^i + x_{t,t-1}^i \end{aligned} \quad (85)$$

is the resource constraint for dynasty i at time t . For $s \geq 1$,

$$\begin{aligned} c_{t+s,t+s}^i + b_{t+s+1}^i &= y_{t+s,t+s}^i + \tau_{t+s}^i \\ c_{t+s,t+s-1}^i + \tau_{t+s}^i &= y_{t+s,t+s-1}^i + Rb_{t+s}^i. \end{aligned}$$

This gives the resource constraint

$$c_{t+s,t+s}^i + c_{t+s,t+s-1}^i + b_{t+s+1}^i = y_{t+s,t+s}^i + y_{t+s,t+s-1}^i + Rb_{t+s}^i \quad (86)$$

Summing over (85) and (86) for $s = 1$ to T , we get

$$\begin{aligned} \sum_{s=0}^T \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i + b_{t+s+1}^i}{R^s} &= y_{t,t}^i + x_{t,t-1}^i + \sum_{s=1}^T \frac{y_{t+s,t+s}^i + y_{t+s,t+s-1}^i + Rb_{t+s}^i}{R^s} \\ \sum_{s=0}^T \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i + b_{t+s+1}^i}{R^s} &= \sum_{s=0}^T \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=1}^T \frac{y_{t+s,t+s-1}^i}{R^s} + x_{t,t-1}^i + \sum_{s=1}^T \frac{b_{t+s}^i}{R^{s-1}} \\ \sum_{s=0}^T \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i + b_{t+s+1}^i}{R^s} &= \sum_{s=0}^T \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=1}^T \frac{y_{t+s,t+s-1}^i}{R^s} + x_{t,t-1}^i + \sum_{s=0}^{T-1} \frac{b_{t+s+1}^i}{R^s} \\ \frac{b_{t+T+1}^i}{R^s} &= \sum_{s=0}^T \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=1}^T \frac{y_{t+s,t+s-1}^i}{R^s} + x_{t,t-1}^i - \sum_{s=0}^T \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i}{R^s} \end{aligned}$$

Using the no-Ponzi game condition, we get as $T \rightarrow \infty$,

$$\sum_{s=0}^{\infty} \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=1}^{\infty} \frac{y_{t+s,t+s-1}^i}{R^s} + x_{t,t-1}^i - \sum_{s=0}^{\infty} \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i}{R^s} \geq 0.$$

Thus we get the dynasty resource constraint

$$\begin{aligned} 0 &\leq \sum_{s=0}^{\infty} \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i}{R^s} \leq \sum_{s=0}^{\infty} \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=1}^{\infty} \frac{y_{t+s,t+s-1}^i}{R^s} + x_{t,t-1}^i \\ 0 &\leq \sum_{s=0}^{\infty} \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i}{R^s} \leq \sum_{s=0}^{\infty} \frac{y_{t+s,t+s}^i}{R^s} + \sum_{s=0}^{\infty} \frac{y_{t+s+1,t+s}^i}{R^{s+1}} + x_{t,t-1}^i \\ 0 &\leq \sum_{s=0}^{\infty} \frac{c_{t+s,t+s}^i + c_{t+s,t+s-1}^i}{R^s} \leq x_{t,t-1}^i + \sum_{s=0}^{\infty} \frac{1}{R^s} \left[y_{t+s,t+s}^i + \frac{y_{t+s+1,t+s}^i}{R} \right] \end{aligned} \quad (87)$$

In the perfect-foresight model, this simplifies to

$$\begin{aligned} 0 &\leq \sum_{s=0}^{\infty} \frac{c_{t+s,t+s} + c_{t+s,t+s-1}}{R^s} \leq x_{t,t-1} + \sum_{s=0}^{\infty} \frac{1}{R^s} \left[y_0 + \frac{y_1}{R} \right] \\ &= x_{t,t-1} + \left[y_0 + \frac{y_1}{R} \right] \frac{1}{1 - \frac{1}{R}} \\ &= x_{t,t-1} + \left[y_0 + \frac{y_1}{R} \right] \frac{R}{r}. \\ 0 &\leq \sum_{s=0}^{\infty} \frac{c_{t+s,t+s} + c_{t+s,t+s-1}}{R^s} \leq x_{t,t-1} + \left[y_0 + \frac{y_1}{R} \right] \frac{R}{r}. \end{aligned} \quad (88)$$

This implies that the minimum cash on hand is

$$x_{t,t-1} \geq x_{\min} = - \left[y_0 + \frac{y_1}{R} \right] \frac{R}{r}. \quad (89)$$

Note that if $x_{t,t-1} = x_{\min}$, we must have $c_{t,t} = c_{t,t-1} = 0$. This implies

$$\tau(x_{\min}) = x_{\min} = - \left[y_0 + \frac{y_1}{R} \right] \frac{R}{r} \quad (90)$$

However, the full resource constraint is only relevant for the paterfamilias and Logan's Run models since the parent or child can demand the other party go along with such a huge transfer from the child to the parent. In the general bargaining model, households must abide by the Aiyagari (1994) constraint that $x \geq 0$ or else the threat point is not defined. Thus

$$b_{t+1}^i \geq - \frac{\min\{y_{t+1,t+1}^i\}}{R}. \quad (91)$$

Since

$$c_{t,t} + b_{t+1} = y_0 + \tau_t,$$

in the perfect-foresight model (91) implies that

$$\begin{aligned} y_0 + \tau_t - c_{t,t} &\geq - \frac{y_1}{R} \\ 0 \leq c_{t,t} &\leq y_0 + \frac{y_1}{R} + \tau_t \end{aligned}$$

further implies that

$$\tau_t \geq - \left[y_0 + \frac{y_1}{R} \right].$$

Thus the transfer is bounded by

$$- \left[y_0 + \frac{y_1}{R} \right] \leq \tau_t \leq x_{t,t-1}. \quad (92)$$

Let us suppose now that $x_2(x_1)$ is a decreasing function of x_1 since a higher x_1 means a higher transfer when the child is the parent. Let $\tilde{\tau}_t(x)$ be the current iteration of the expected transfer function and let $\tau_t(x)$ be the resulting iteration of the actual transfer function. Suppose that $\tilde{\tau}_t$ and τ_t are increasing in x . The (3) implies the next iteration of $\tilde{\tau}$ is

$$\begin{aligned} \tilde{\tau}_{t+1}(x) &= \tau_t(y_1 + Rb(\tau_t(x), \tilde{\tau}_{t+1}(x))) \\ &= \tau_t \left(y_1 + R \frac{1}{1+\phi} \left[y_0 + \tau_t(x) - \frac{\phi}{R}(y_1 - \tilde{\tau}_t(x)) \right] \right) \end{aligned}$$

$$\tilde{\tau}'_{t+1}(x) = \tau'_t \left(y_1 + R \frac{1}{1+\phi} \left[y_0 + \tau_t(x) - \frac{\phi}{R}(y_1 - \tilde{\tau}_t(x)) \right] \right) \frac{R}{1+\phi} \left(\tau'_t(x) + \frac{\phi}{R} \tilde{\tau}'_t(x) \right) \geq 0.$$

Thus if we start out with a guess that the expected transfer function is increasing, and the resulting transfer function is increasing, the expected transfer function will remain increasing.

Then we have

$$\begin{aligned} &([u(x_1 - \tau) - u(x_1)] + (\theta\eta_2 + (1-\theta)\eta_1)[u(x_2 + \tau) - u(x_2)])(x_1 - \tau)^{-\gamma} \\ &= (((1-\theta)\eta_2 + \theta\eta_1)[u(x_1 - \tau) - u(x_1)] + \eta_1\eta_2[u(x_2 + \tau) - u(x_2)])(x_2 + \tau)^{-\gamma} \end{aligned}$$

If $\eta_1 = \eta_2 = \eta$, this simplifies to

$$\begin{aligned} (x_1 - \tau)^{-\gamma} &= \eta(x_2 + \tau)^{-\gamma} \\ x_1 - \tau &= \eta^{-1/\gamma}(x_2 + \tau) \end{aligned}$$

$$\begin{aligned}
dx_1 - d\tau &= \eta^{-1/\gamma} \left(\frac{dx_2}{dx_1} dx_1 + d\tau \right) \\
\left(1 - \eta^{-1/\gamma} \frac{dx_2}{dx_1} \right) dx_1 &= (1 + \eta^{-1/\gamma}) d\tau \\
\frac{d\tau}{dx_1} &= \frac{1 - \eta^{-1/\gamma} \frac{dx_2}{dx_1}}{1 + \eta^{-1/\gamma}} > 0
\end{aligned}$$

since $dx_2/dx_1 \leq 0$.

$$\begin{aligned}
& \left[(x_1 - \tau)^{-\gamma} - x_1^{-\gamma} + (\theta\eta_2 + (1 - \theta)\eta_1)[(x_2 + \tau)^{-\gamma} - x_2^{-\gamma}] \frac{dx_2}{dx_1} \right] (x_1 - \tau)^{-\gamma} dx_1 \\
& - \gamma ([u(x_1 - \tau) - u(x_1)] + (\theta\eta_2 + (1 - \theta)\eta_1)[u(x_2 + \tau) - u(x_2)]) (x_1 - \tau)^{-\gamma-1} (dx_1 - d\tau) \\
& + [-(x_1 - \tau)^{-\gamma} + (\theta\eta_2 + (1 - \theta)\eta_1)(x_2 + \tau)^{-\gamma}] (x_1 - \tau)^{-\gamma} d\tau \\
= & \left[((1 - \theta)\eta_2 + \theta\eta_1)[(x_1 - \tau)^{-\gamma} - x_1^{-\gamma}] + \eta_1\eta_2[(x_2 + \tau)^{-\gamma} - x_2^{-\gamma}] \frac{dx_2}{dx_1} \right] (x_2 + \tau)^{-\gamma} dx_1 \\
& - \gamma (((1 - \theta)\eta_2 + \theta\eta_1)[u(x_1 - \tau) - u(x_1)] + \eta_1\eta_2[u(x_2 + \tau) - u(x_2)]) (x_2 + \tau)^{-\gamma-1} \left(\frac{dx_2}{dx_1} dx_1 + d\tau \right) \\
& + [-((1 - \theta)\eta_2 + \theta\eta_1)(x_1 - \tau)^{-\gamma} + \eta_1\eta_2(x_2 + \tau)^{-\gamma}] (x_2 + \tau)^{-\gamma} d\tau
\end{aligned}$$

Note that if $dx_2/dx_1 \leq 0$,

$$\frac{d\tau_1^*}{dx_1} = \frac{1 - \eta_1^{-1/\gamma} \frac{dx_2}{dx_1}}{1 + \eta_1^{-1/\gamma}} > 0$$

Likewise,

$$\frac{d\tau_2^*}{dx_1} = \frac{1 - \eta_2^{-1/\gamma} \frac{dx_2}{dx_1}}{1 + \eta_2^{-1/\gamma}} > 0$$

$$\begin{aligned}
U_1(\tau|x_1) &= A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2(x_1) + \tau) - u(x_2(x_1))] \\
U_2(\tau|x_1) &= A_{21}[u(x_1 - \tau) - u(x_1)] + A_{22}[u(x_2(x_1) + \tau) - u(x_2(x_1))]
\end{aligned}$$

Let us assume that $x_2(x_1)$ is decreasing. There will be x_1^- and x_1^+ such that

$$\frac{x_2(x_1^-)}{x_1^-} = \max\{\eta_1^{1/\gamma}, \eta_2^{1/\gamma}\} \quad (93)$$

and

$$\frac{x_2(x_1^+)}{x_1^+} = \min\{\eta_1^{1/\gamma}, \eta_2^{1/\gamma}\}. \quad (94)$$

$$\begin{aligned}
0 = U'(\tau|x_1) &= \theta \frac{-A_{11}u'(x_1 - \tau) + A_{12}u'(x_2 + \tau)}{A_{11}[u(x_1 - \tau) - u(x_1)] + A_{12}[u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
& + (1 - \theta) \frac{-A_{21}u'(x_1 - \tau) + A_{22}u'(x_2 + \tau)}{A_{21}[u(x_1 - \tau) - u(x_1)] + A_{22}[u(x_2(x_1) + \tau) - u(x_2(x_1))]}
\end{aligned}$$

$$0 = U'(\tau|x_1) = \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \quad (95)$$

$$+ (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \quad (96)$$

Since this must hold for all x_1 ,

$$U''(\tau|x_1)d\tau + \frac{\partial U'(\tau|x_1)}{\partial x_1} dx_1 = 0$$

$$\frac{d\tau}{dx_1} = - \frac{U''(\tau|x_1)}{\frac{\partial U'(\tau|x_1)}{\partial x_1}}.$$

$$\begin{aligned} U''(\tau|x_1) &= \theta \frac{u''(x_1 - \tau) + \eta_1 u''(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\ &+ (1 - \theta) \frac{u''(x_1 - \tau) + \eta_2 u''(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\ &- \theta \frac{(-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau))^2}{([u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \\ &- (1 - \theta) \frac{(-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau))^2}{([u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \\ &< 0 \end{aligned}$$

as long as $U_1(\tau|x_1), U_2(\tau|x_2) > 0$.

$$\begin{aligned} \frac{\partial U'(\tau|x_1)}{\partial x_1} &= \theta \frac{-u''(x_1 - \tau) + \eta_1 u''(x_2 + \tau) \frac{dx_2}{dx_1}}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\ &+ (1 - \theta) \frac{-u''(x_1 - \tau) + \eta_2 u''(x_2 + \tau) \frac{dx_2}{dx_1}}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\ &- \theta \frac{(-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_1 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))] \frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \\ &- (1 - \theta) \frac{(-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_2 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))] \frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \end{aligned}$$

The first two terms are both positive since u is strictly concave and $dx_2/dx_1 < 0$. Empirically, I find in “bargainingpfderivative.nb” that the other two terms may not be positive, although so far I have only found cases where the sum is positive.

Using (95), these terms are

$$\begin{aligned}
& -\theta \frac{(-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_1 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \\
& -(1 - \theta) \frac{(-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_2 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))])^2} \\
= & (1 - \theta) \frac{(-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_1 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]) ([u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))])} \\
& + \theta \frac{(-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_2 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right)}{([u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]) ([u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))])}
\end{aligned}$$

Assuming $U_1(\tau), U_2(\tau) > 0$, the denominator is positive. The numerator is

$$\begin{aligned}
N &= (1 - \theta) (-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_1 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right) \\
& \quad \theta (-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)) \left(u'(x_1 - \tau) - u'(x_1) + \eta_2 [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \right)
\end{aligned}$$

Let

$$N_1 = [(1 - \theta) (-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) + \theta (-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau))] [u'(x_1 - \tau) - u'(x_1)] \quad (97)$$

and

$$\begin{aligned}
N_2 &= (1 - \theta) \eta_1 (-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)) [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1} \\
& \quad + \theta \eta_2 (-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)) [u'(x_2(x_1) + \tau) - u'(x_2(x_1))]\frac{dx_2}{dx_1}, \quad (98)
\end{aligned}$$

so $N = N_1 + N_2$.

$$N_1 = (-u'(x_1 - \tau) + (\theta \eta_1 + (1 - \theta) \eta_2)) u'(x_2 + \tau) [u'(x_1 - \tau) - u'(x_1)]. \quad (99)$$

If $\tau > 0$, the second factor is positive because u' is strictly decreasing.

$$\begin{aligned}
0 &= \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
& \quad + (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]}
\end{aligned}$$

Since the denominators are both positive, one numerator must be positive and the other negative. Suppose $\eta_1 < \eta_2$. Since $u' > 0$, we must have

$$-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau) < 0 < -u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau). \quad (100)$$

Then

$$\begin{aligned}
0 &= \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&\quad + (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&< \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&\quad + (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} .
\end{aligned}$$

Since u is strictly increasing and $\tau > 0$, $u(x_2(x_1) + \tau) - u(x_2(x_1)) > 0$, so $[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))] > [u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]$. Thus

$$\begin{aligned}
0 &< \frac{-u'(x_1 - \tau) + (\theta\eta_1 + (1 - \theta)\eta_2)u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&0 < -u'(x_1 - \tau) + (\theta\eta_1 + (1 - \theta)\eta_2)u'(x_2 + \tau),
\end{aligned}$$

and so $N_1 > 0$. The analogous argument holds if $\eta_1 > \eta_2$.

Alternatively, if $\tau < 0$, the second factor in (99) is negative and $u(x_2(x_1) + \tau) - u(x_2(x_1)) > 0$. If $\eta_1 < \eta_2$, we must still have (100). Then since $\tau < 0$, $u(x_2(x_1) + \tau) - u(x_2(x_1)) < 0$, so $[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))] < [u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]$. Therefore

$$\begin{aligned}
0 &= \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&\quad + (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_2 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&> \theta \frac{-u'(x_1 - \tau) + \eta_1 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} \\
&\quad + (1 - \theta) \frac{-u'(x_1 - \tau) + \eta_2 u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} ,
\end{aligned}$$

so

$$0 > \frac{-u'(x_1 - \tau) + (\theta\eta_1 + (1 - \theta)\eta_2)u'(x_2 + \tau)}{[u(x_1 - \tau) - u(x_1)] + \eta_1 [u(x_2(x_1) + \tau) - u(x_2(x_1))]} .$$

Since the denominator is still positive,

$$0 > -u'(x_1 - \tau) + (\theta\eta_1 + (1 - \theta)\eta_2)u'(x_2 + \tau),$$

so $N_1 > 0$ again. Thus if x_2 is independent of x_1 , we must have $\tau(x_1)$ an increasing function.

$$N_2 = [((1 - \theta)\eta_1 + \theta\eta_2)(-u'(x_1 - \tau) + \eta_1\eta_2 u'(x_2 + \tau))] [u'(x_2(x_1) + \tau) - u'(x_2(x_1))] \frac{dx_2}{dx_1}.$$

If $\frac{dx_2}{dx_1} < 0$, $[u'(x_2(x_1) + \tau) - u'(x_2(x_1))] \frac{dx_2}{dx_1}$ will have the same sign as $u'(x_1 - \tau) - u'(x_1)$. To show that $N_2 > 0$, we need to show that

$$DMU_h = -u'(x_1 - \tau) + \frac{\eta_1\eta_2}{(1 - \theta)\eta_1 + \theta\eta_2} u'(x_2 + \tau)$$

has the same sign as

$$DMU_a - u'(x_1 - \tau) + (\theta\eta_1 + (1 - \theta)\eta_2)u'(x_2 + \tau),$$

where a stands for arithmetic mean and h stands for harmonic mean since we can rewrite

$$DMU_h = -u'(x_1 - \tau) + \frac{1}{\frac{\theta}{\eta_1} + \frac{(1-\theta)}{\eta_2}} u'(x_2 + \tau).$$

We can rewrite the Nash condition as

$$\begin{aligned} & ([u(x_1 - \tau) - u(x_1)] + (\theta\eta_2 + (1 - \theta)\eta_1)[u(x_2 + \tau) - u(x_2)]) u'(x_1 - \tau) \\ = & (((1 - \theta)\eta_2 + \theta\eta_1)[u(x_1 - \tau) - u(x_1)] + \eta_1\eta_2[u(x_2 + \tau) - u(x_2)]) u'(x_2 + \tau) \\ & - [u'(x_1 - \tau) - ((1 - \theta)\eta_2 + \theta\eta_1)u'(x_2 + \tau)][u(x_1 - \tau) - u(x_1)] \\ = & ((1 - \theta)\eta_1 + \theta\eta_2) \left[u'(x_1 - \tau) - \frac{\eta_1\eta_2}{(1 - \theta)\eta_1 + \theta\eta_2} u'(x_2 + \tau) \right] [u(x_2 + \tau) - u(x_2)] \end{aligned}$$

Since u is strictly increasing, $u(x_2 + \tau) - u(x_2)$ and $u(x_1) - u(x_1 - \tau)$ must have the same sign. Consequently DMU_a and DMU_h also have the same sign.

Let $f(x)$ be an expected transfer function. Let $\tau_f(x)$ be the optimal transfer given the expected transfer function f . The mapping to determine the next iteration of the expected transfer function is

$$\begin{aligned} (Tf)(x) &= \tau_f(y_1 + Rb_{t+1}(\tau_f(x), f(x))) \\ (Tf)(x) &= \tau_f \left(y_1 + R \frac{1}{1 + \phi} \left[y_0 + \tau_f(x) - \frac{\phi}{R}(y_1 - f(x)) \right] \right) \\ (Tf)(x) &= \tau_f \left(R \frac{1}{1 + \phi} \left[\frac{(1 + \phi)y_1}{R} + y_0 + \tau_f(x) - \frac{\phi}{R}y_1 + \frac{\phi}{R}f(x) \right] \right) \\ (Tf)(x) &= \tau_f \left(R \frac{1}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \tau_f(x) + \frac{\phi}{R}f(x) \right] \right) \\ (Tf)(x) &= \tau_f \left(R \frac{1}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \tau_f(x) \right] + \frac{\phi}{1 + \phi} f(x) \right) \end{aligned} \tag{101}$$

The induced transfer function is

$$\begin{aligned} \tau_f(x) &= \arg \max_{\tau \in -[y_0 + \frac{y_1}{R}, x]} \left(\max \left\{ u(x - \tau) - u(\tau) + \eta_p \left[u \left(y_0 + \frac{y_1 - f(x)}{R} + \tau \right) - u \left(y_0 + \frac{y_1 - f(x)}{R} \right) \right], 0 \right\} \right)^\theta \\ &\quad \times \left(\max \left\{ u(x - \tau) - u(\tau) + \eta_c \left[u \left(y_0 + \frac{y_1 - f(x)}{R} + \tau \right) - u \left(y_0 + \frac{y_1 - f(x)}{R} \right) \right], 0 \right\} \right)^{1-\theta} \end{aligned} \tag{102}$$

We have already shown that τ_f is increasing in x . Since the model is symmetric between player 1 (i.e. the parent) and 2 (i.e. the child), the flip-side of that argument is that τ will be decreasing in the child's wealth. Consequently, we will have $\tau_f(x) \leq \tau_g(x)$ for all x if $f(x) \leq g(x)$ for all x . Then

$$\begin{aligned} (Tf)(x) &= \tau_f \left(R \frac{1}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \tau_f(x) \right] + \frac{\phi}{1 + \phi} f(x) \right) \leq \tau_f \left(R \frac{1}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \tau_g(x) \right] + \frac{\phi}{1 + \phi} g(x) \right) \\ &\leq \tau_g \left(R \frac{1}{1 + \phi} \left[y_0 + \frac{y_1}{R} + \tau_g(x) \right] + \frac{\phi}{1 + \phi} g(x) \right) = (Tg)(x). \end{aligned}$$

That establishes the first Blackwell condition. If we can further show the second Blackwell condition, we will establish that T is a contraction. Empirically, I do find that the discounting condition holds.

A problem does arise when $f(x) > Ry_0 + y_1$ because then the expected transfer cannot be made if there is no bargaining. We need to adjust the threat point so

$$x_2 = y_0 + \frac{y_1 - f(x)}{R} + B(x) \geq 0.$$

If $\gamma \geq 1$, as $x_2 \rightarrow 0$, the bargaining condition becomes

$$\begin{aligned} & \lim_{x_2 \rightarrow 0} ([u(x_1 - \tau) - u(x_1)] + (\theta\eta_2 + (1 - \theta)\eta_1)[u(x_2 + \tau) - u(x_2)])(x_1 - \tau)^{-\gamma} \\ = & \lim_{x_2 \rightarrow 0} (((1 - \theta)\eta_2 + \theta\eta_1)[u(x_1 - \tau) - u(x_1)] + \eta_1\eta_2[u(x_2 + \tau) - u(x_2)])(x_2 + \tau)^{-\gamma}, \\ & (\theta\eta_2 + (1 - \theta)\eta_1)(x_1 - \tau)^{-\gamma} = \eta_1\eta_2\tau^{-\gamma}. \\ & x_1 - \tau = \left(\frac{\theta\eta_2 + (1 - \theta)\eta_1}{\eta_1\eta_2} \right)^{\frac{1}{\gamma}} \tau \\ & x_1 - \tau = \left(\frac{\theta}{\eta_1} + \frac{1 - \theta}{\eta_2} \right)^{\frac{1}{\gamma}} \tau \\ & \tau = \frac{x_1}{1 + \left(\frac{\theta}{\eta_1} + \frac{1 - \theta}{\eta_2} \right)^{\frac{1}{\gamma}}} \end{aligned} \quad (103)$$

When $\gamma < 1$, we can still solve (84) for τ .

One of the motivating questions of the paper was to understand under what circumstances the model will behave like a standard overlapping-generations model without transfers. If there is no transfer,

$$b = \frac{1}{1 + \phi} \left[y_0 - \frac{\phi}{R} y_1 \right]. \quad (104)$$

The parent's wealth will then be

$$x_p = y_1 + Rb = y_1 + \frac{R}{1 + \phi} \left[y_0 - \frac{\phi}{R} y_1 \right] = \frac{R}{1 + \phi} \left[\frac{1 + \phi}{R} y_1 + y_0 - \frac{\phi}{R} y_1 \right] = \frac{R}{1 + \phi} \left[y_0 + \frac{y_1}{R} \right]. \quad (105)$$

The child's wealth will be

$$x_c = y_0 + \frac{y_1}{R} \quad (106)$$

Thus

$$\frac{x_c}{x_p} = \frac{1 + \phi}{R} \quad (107)$$

The condition for this no transfer equilibrium to hold will then be

$$\min\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\} \leq \frac{1 + \phi}{R} \leq \max\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\}. \quad (108)$$

If we set $\beta = .97^{30}$, $\alpha = 1/3$, $\gamma = 1$, $\delta = 1$, $e_0 = 1$, and $e_1 = 1/3$, we will have $K/Y = 3$ years, $\phi = 2.49$ and $R = 3.12$ (equivalent to $r = 3.9\%$). Then

$$\frac{1 + \phi}{R} = 1.17.$$

Thus if $\eta_c \leq 1.17 \leq \eta_p$ we will have a no-transfer equilibrium. (“twoperiodolg.xls”) Hong and Rios-Rull use life-insurance data to estimate $\eta_c = 3.5$. If we assume children care about parent’s consumption at least as much as their own, then we should see no transfers in the steady state. However, if $\eta_p = 1$, then an exogenous transfer from the child to the parent of more than 9% of the child’s lifetime wealth ought to lead the parents to make an opposing transfer. Note that $\eta_p \leq 1$ means that children value parental consumption at least as much as their own. Note that 1.17 is a lower bound on η_p under which no transfer ought to occur. This may help to explain why so many tests of bequest models have failed. If children do not care about parents, as most models assume, we ought to see large transfers from parents to children, even in the steady state.

Note that if we have an equilibrium where $\tau = 0$ when $x_p = \frac{R}{1+\phi} [y_0 + \frac{y_1}{R}]$, then for any parental wealth under which there is no transfer, the child will expect to make no transfer. Let us define x_t^p to be the time series of the parent’s wealth. Then $x_t^p = \frac{R}{1+\phi} [y_0 + \frac{y_1}{R}]$ is an absorbing state. Consequently, if we have

$$\min\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\} \leq \frac{y_0 + \frac{y_1}{R}}{x_t^p} \leq \max\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\}$$

there will be no bargaining so the range in which no transfer occurs is

$$\frac{y_0 + \frac{y_1}{R}}{\max\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\}} \leq x_t^p \leq \frac{y_0 + \frac{y_1}{R}}{\min\{\eta_p^{1/\gamma}, \eta_c^{1/\gamma}\}}. \quad (109)$$

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