Log-Linearization Methods in OLG Models with an Application to Social Security

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Overview

- Build an OLG model with relatively short periods
- Model has:
  - demographic dynamics from exogenous birth rates, death rates and immigration
  - aggregate productivity shocks, but no idiosyncratic shocks to individuals within cohorts
- Calibrate the model to the US economy and Social Security system.
- Solve the model using linearization techniques commonly applied to DSGE models.
- Using Monte Carlo methods, generate forecasts of the balance for the Social Security trust fund with confidence bands corresponding to the uncertainty associated with business cycle & demographic fluctuations.

Questions & Motivation

When will the social security trust fund run out?

How do you solve DSGE models that are not stationary?

Large amount of current research on countercyclical fiscal policy and reducing national debt.

Two biggest problems for U.S. national debt (Soc. Sec. and Medicare/Medicaid) require OLG modeling.

Need a solution for a nonstationary OLG model.

The Model – Demographics

Cohorts decrease in size over time due to deaths, but also increase in size due to immigration.

$$N'_{s+1} = N_s(\rho_{s+1} + \iota_{s+1}) \text{ for } 1 \leq s \leq S - 1 \qquad (2.1)$$

New births each period are the fertility rate per period for each age cohort times the number of people in the cohort.

$$N'_1 = \sum_{s=1}^{S} f_s N_s \qquad (2.2)$$

The Model – Households

Each period households receive wage income $(w\bar{\ell}_s)$, interest income$[(1 + r - \delta)k_s]$, lump-sum transfers (T), and social security benefits (b_s).

They use these fund to pay taxes $(\tau w\bar{\ell}_s)$, and to purchase new capital (k'_{s+1}) and consumption (c_s).

The household's problem is written using a Bellman equation:

$$V_s(k_s, \Omega) = \max_{k_{s+1}'} u\{c_s\} + \beta\rho_{s+1}E\{V_{s+1}(\Omega', k'_{s+1})\}$$

$$c_s = w\bar{\ell}_s(1 - \tau) + (1 + r - \delta)k_s - k'_{s+1} + b_s + T \qquad (2.3)$$

This problem yields the following Euler equation for a household of age s:

$$u^c\{c_s\} = \beta\rho_{s+1}E\{u^c\{c'_s\}(1 + r' - \delta)\} \qquad (2.4)$$

The Model – Firms

Firms hire labor and capital to maximize profits each period.

$$\max_{K,L} K^{\alpha}(e^{gt+z}L)^{1-\alpha} - rK - wL$$

The solution is characterized by the following three equations.

$$r = \alpha Y/K \tag{2.5}$$

$$w = (1-\alpha)Y/L \tag{2.6}$$

$$Y = K^{\alpha}(e^{gt+z}L)^{1-\alpha} \tag{2.7}$$

The Model – Stochastic Processes

Technology is assumed to evolve over time according to the following law of motion.

$$z' = \psi_z z + e_z{}'; \; e_z{}' \sim iid(0, \sigma_z^2) \qquad (2.8)$$

The Model - Government

The government accumulates a balance over time on a trust fund.

$$H' = H + \sum_{s=E}^{R-1} N_s \tau w \bar{\ell}_s - \sum_{s=R}^{S} N_s b_s \tag{2.10}$$

AIME evolves over ages E to R according to:

$$a'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1} + \iota_{s+1}} \left[\frac{s-E-1}{s-E} a_s + \frac{1}{s-E} w \bar{\ell}_s \right] \text{ for } E \leq s \leq R-1 \tag{2.13}$$

Benefits are assigned when a household retires at age R and are a function of AIME at retirement.

$$b_R = \theta a_R \tag{2.11}$$

Once set at retirement benefits remain constant until death.

$$b'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1} + \iota_{s+1}} b_s \text{ for } s > R \tag{2.12}$$

The Model – Bequests

We model redistribution of the capital of deceased households over the current population, by assuming an equal share for each household regardless of age.

$$T' = \frac{\sum_{s=1}^{S} N_s(1-\rho_s)k_s}{\sum_{s=1}^{S} N'_s} \quad (2.9)$$

The Model – Market Clearing

The capital and labor market clearing conditions are given by:

$$K = \sum_{s=1}^{S} N_s k_s + H \tag{2.14}$$

$$L = \sum_{s=1}^{S} N_s \bar{\ell}_s \tag{2.15}$$

There is also a goods market clearing condition

$$Y + (1 - \delta)K = \sum_{s=1}^{S} c_s + K',$$

but it is redundant by Walras Law.

The Model - Stationarizing

We must transform the non-stationary variables to stationary ones, denoted with a carat (^).

Some per capita variables, such as consumption and wages, will grow at the long-run rate of g.

$\hat{x} \equiv x/e^{gt}$ for $x \in (\{k_s\}_{s=2}^{S}, \{a_s\}_{s=E}^{R}, \{b_s\}_{s=R}^{S}, \{c_s\}_{s=1}^{S}, w)$

To transform cohort populations we need to remove a unit root, which we do by dividing by the total population, N.

$\hat{x} \equiv x/N$ for $x \in (\{N_s\}_{s=1}^{S}, L)$

Some aggregate variables grow at the rate g and also have a unit root.

$\hat{x} \equiv x/(Ne^{gt})$ for $x \in (Y, K, H)$

Calibration

S	maximum age in periods
E	period workers enter the labor force
R	period workers retire
$\{\bar{\ell}_s\}_{s=1}^{S}$	effective labor by age
$\{\bar{f}_s\}_{s=1}^{S}$	average fertility rates by age
$\{\bar{\iota}_s\}_{s=2}^{S}$	average immigration rates by age
$\{\bar{\rho}_s\}_{s=2}^{S}$	average survival rates by age
τ	payroll tax rate
δ	capital depreciation rate
β	subjective discount factor
g	growth rate of technology
γ	coefficient of relative risk aversion
α	capital share in GDP
θ	pension benefits as percent of AIME

In addition we have parameters governing the stochastic processes

$\psi_z, \{\psi_{fs}, \psi_{\iota s}, \psi_{\rho s}\}_{s=1}^{S}$	autocorrelations
$\sigma_z^2, \{\sigma_{fs}^2, \sigma_{\iota s}^2, \sigma_{\rho s}^2\}_{s=1}^{S}$	variances

Calibration

$E = 16$	age of entry into the labor force in years
$R = 65$	age of retirement in years
$\delta = .10$	capital depreciation rate per year
$\beta = .98$	subjective discount factor per year
$g = .005$	growth rate of technology per year
$\gamma = 1.0$	coefficient of relative risk aversion
$\alpha = .33$	capital share in GDP
$\theta = .20$	pension benefits as percent of AIME
$\tau = .0392$	payroll tax rate
$\psi_z = .90$	autocorrelation of technology (per year)
$\sigma_z^2 = .0004$	variance of technology

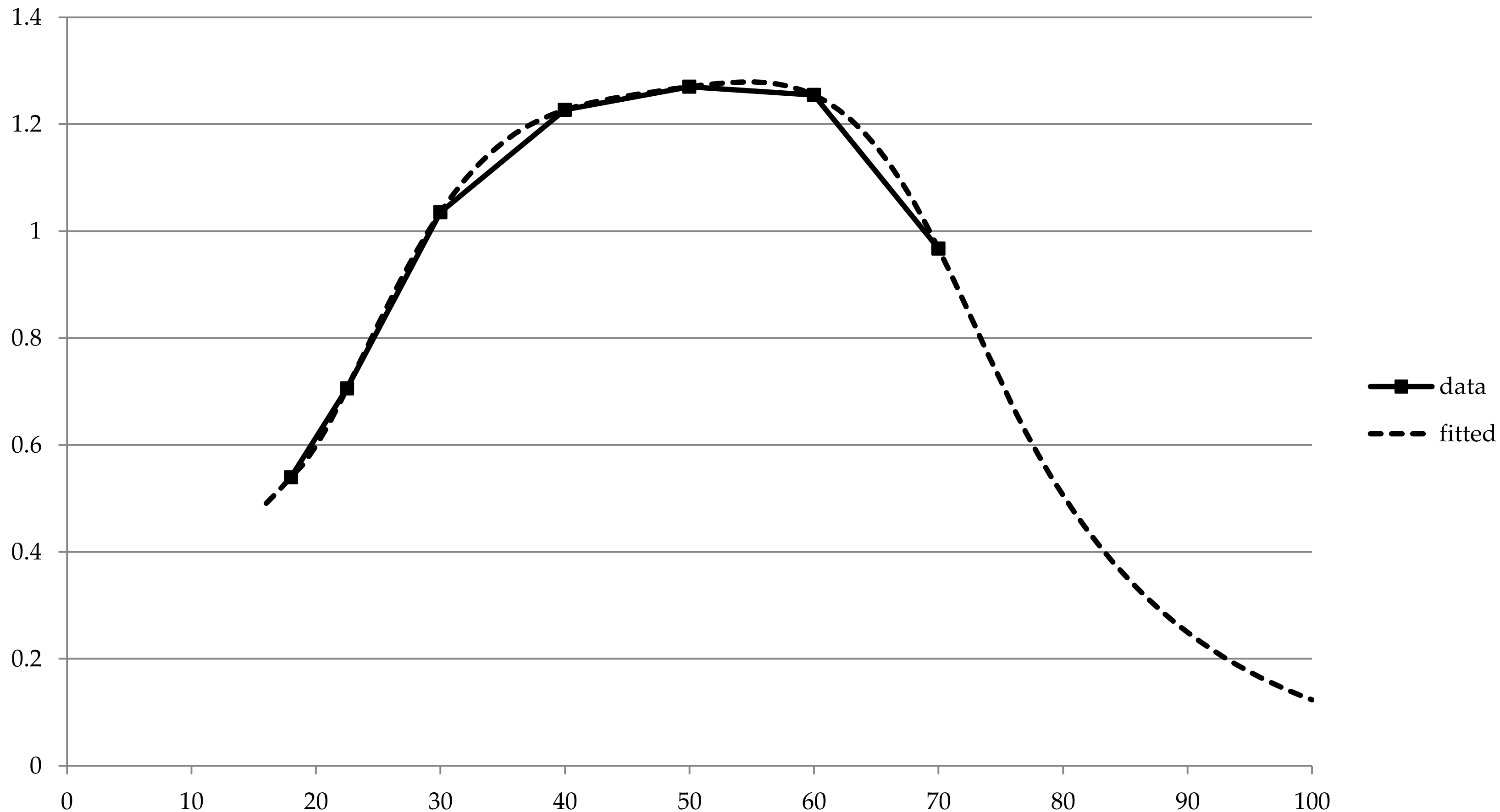
Calibration

Average monthly OASDI only benefit to earnings ratio

2001	18.66%
2002	18.75%
2003	18.34%
2004	18.20%
2005	18.30%
2006	18.42%
2007	18.58%
2008	18.56%
2009	19.56%
average	18.60%

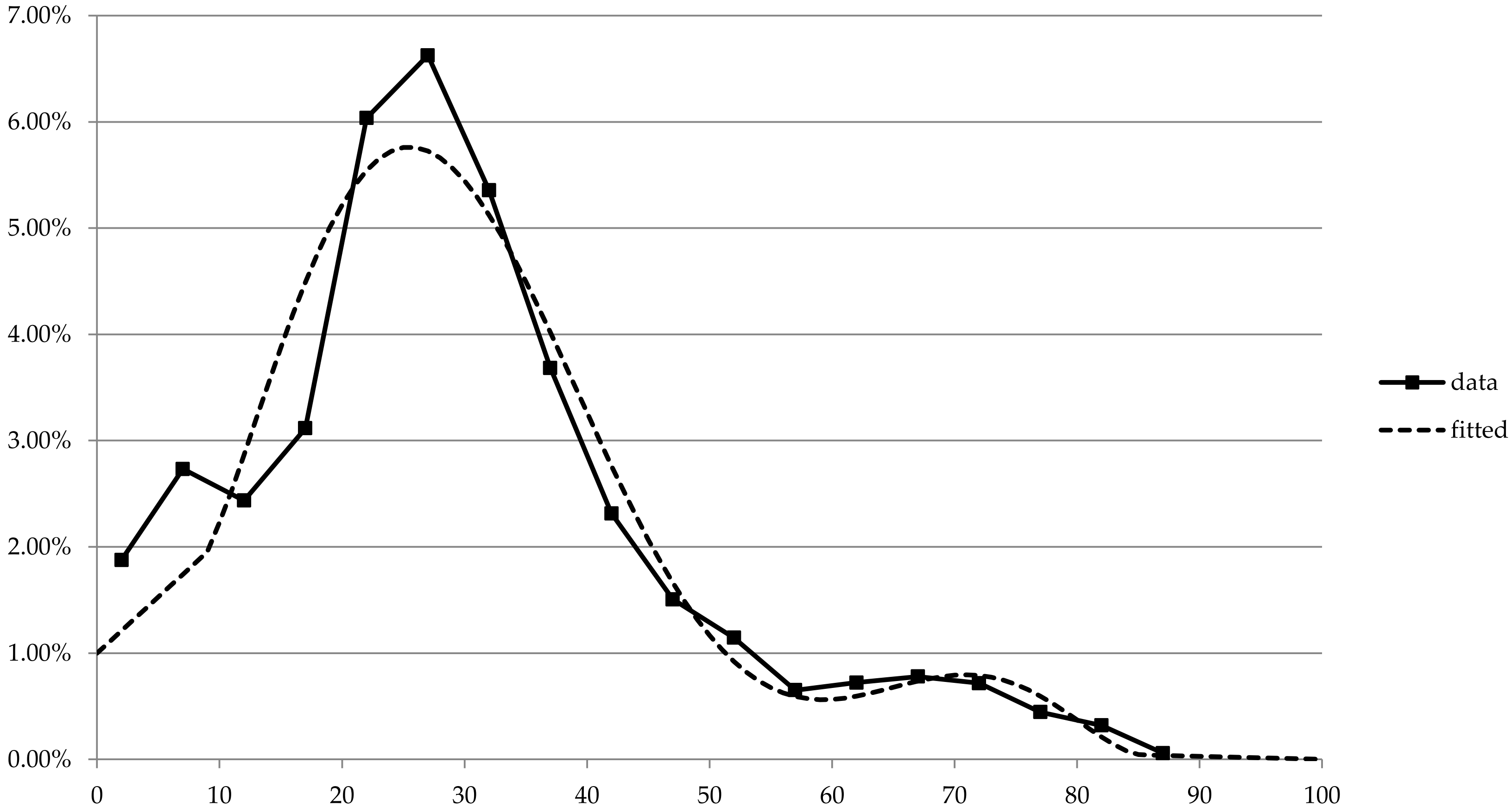
Calibration

Fit a polynomial to effective labor by age



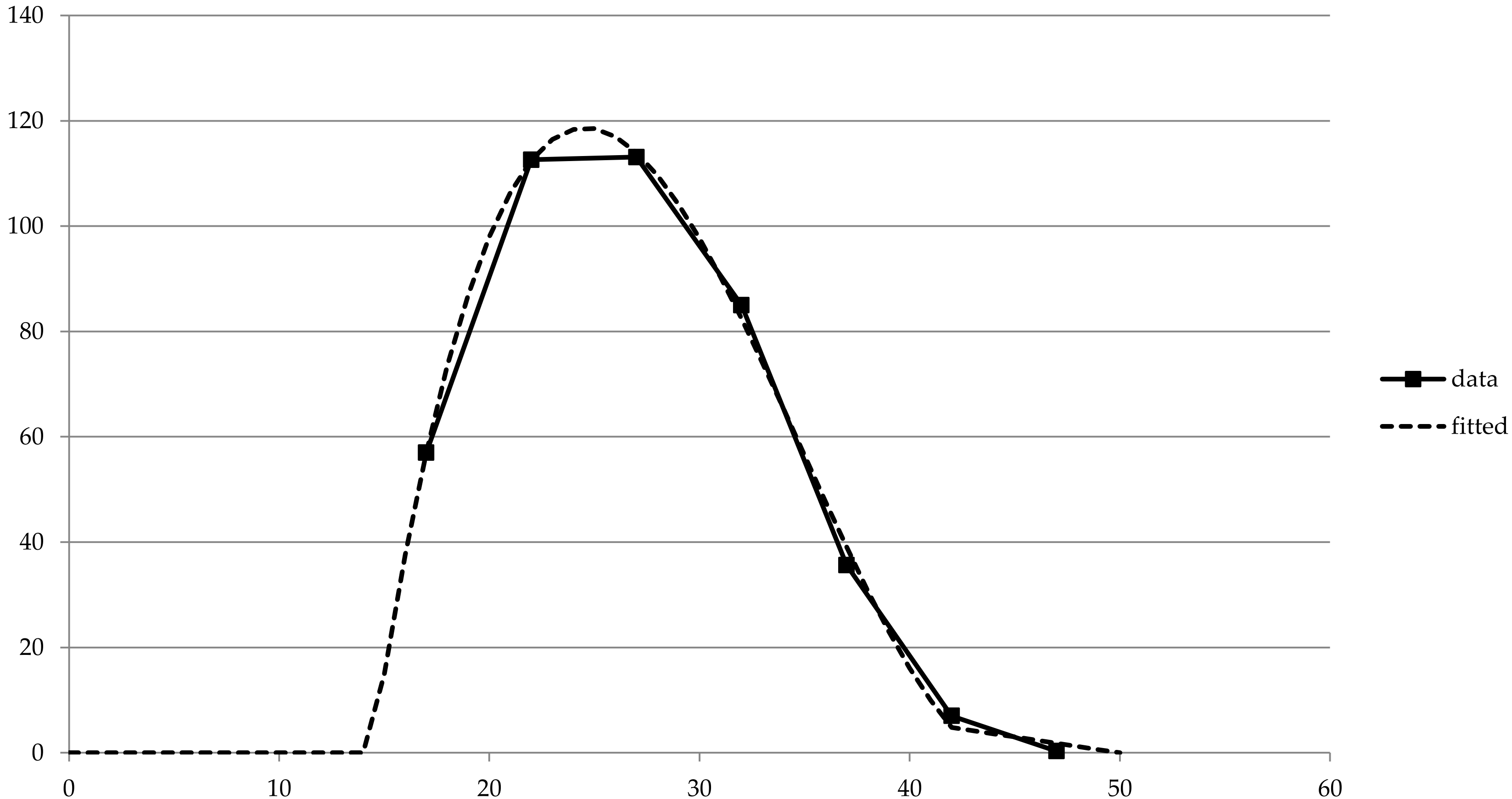
Calibration

Fit a polynomial to immigration rates by age



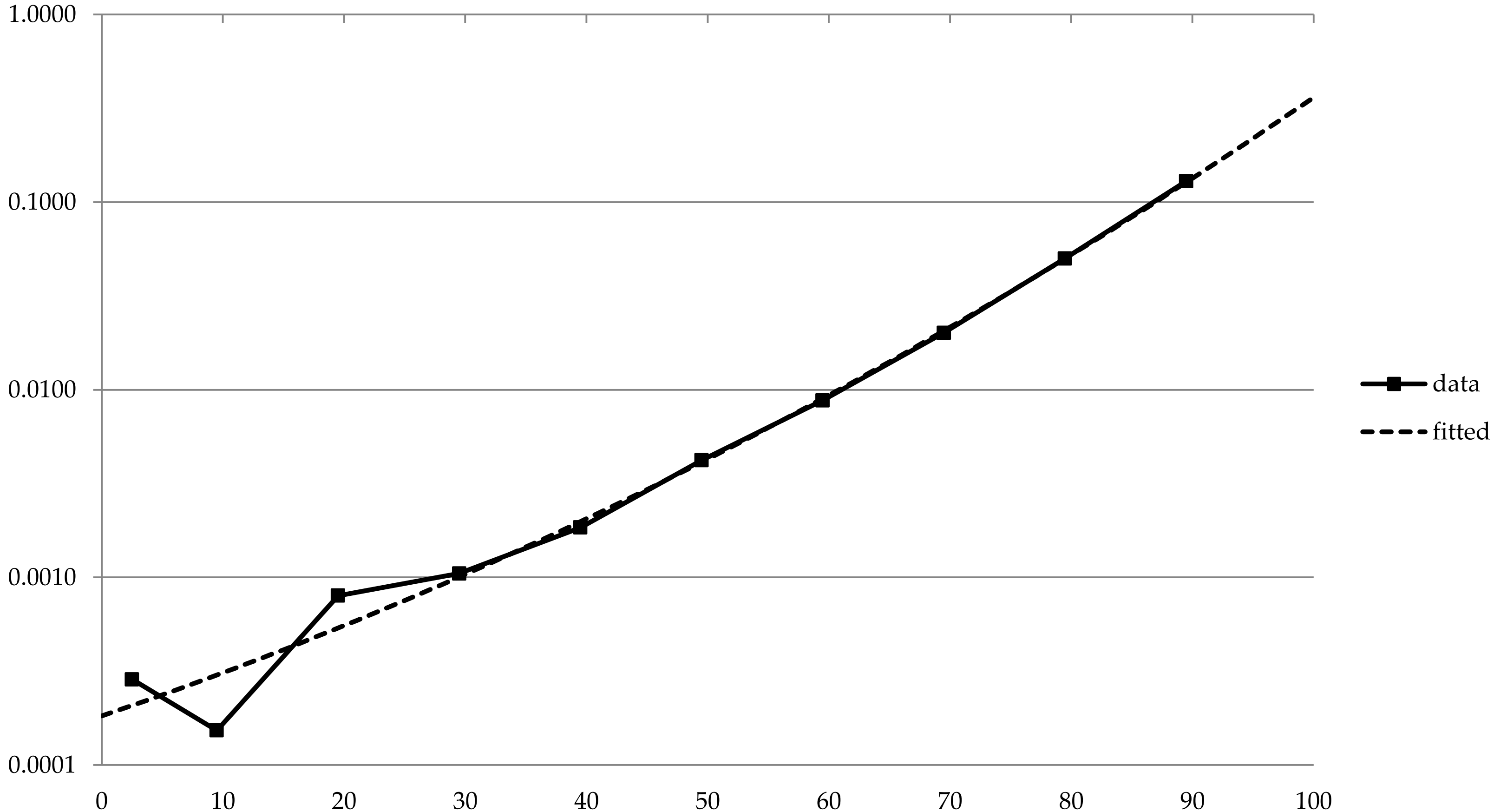
Calibration

Fit a polynomial to fertility rates by age



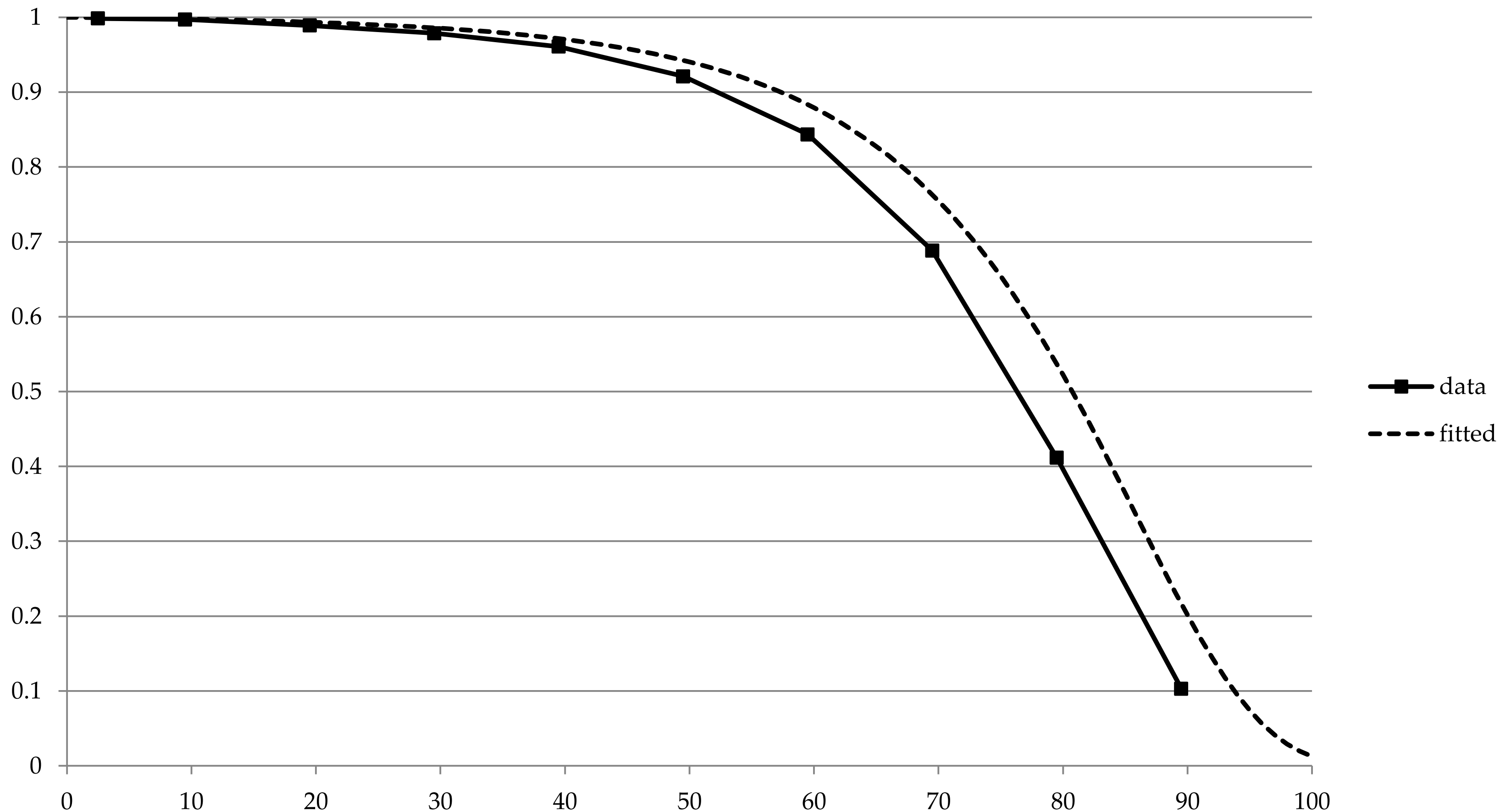
Calibration

Fit a polynomial to death hazard rates by age (log scale)



Calibration

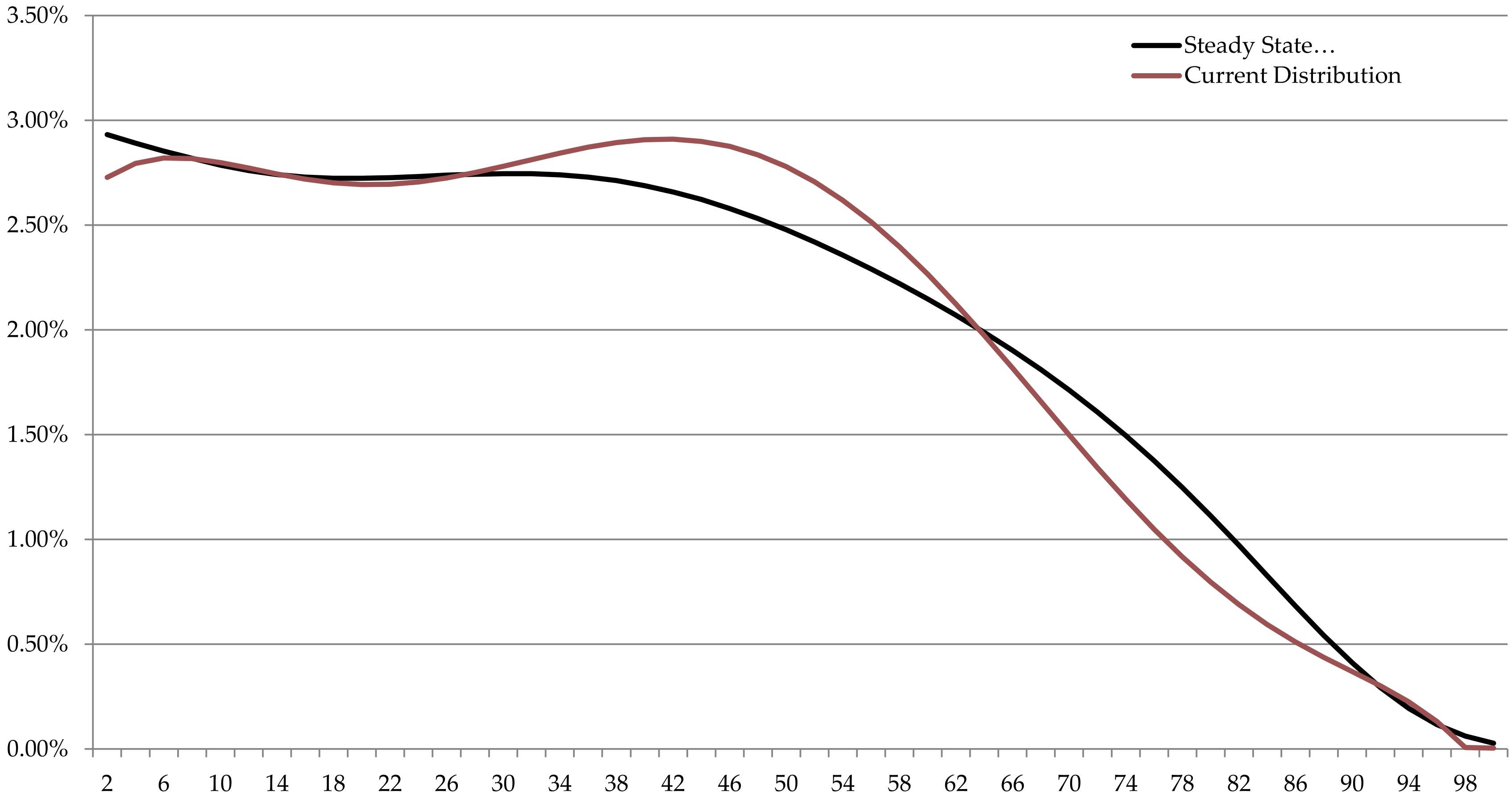
- Implied cumulative survival rates by age



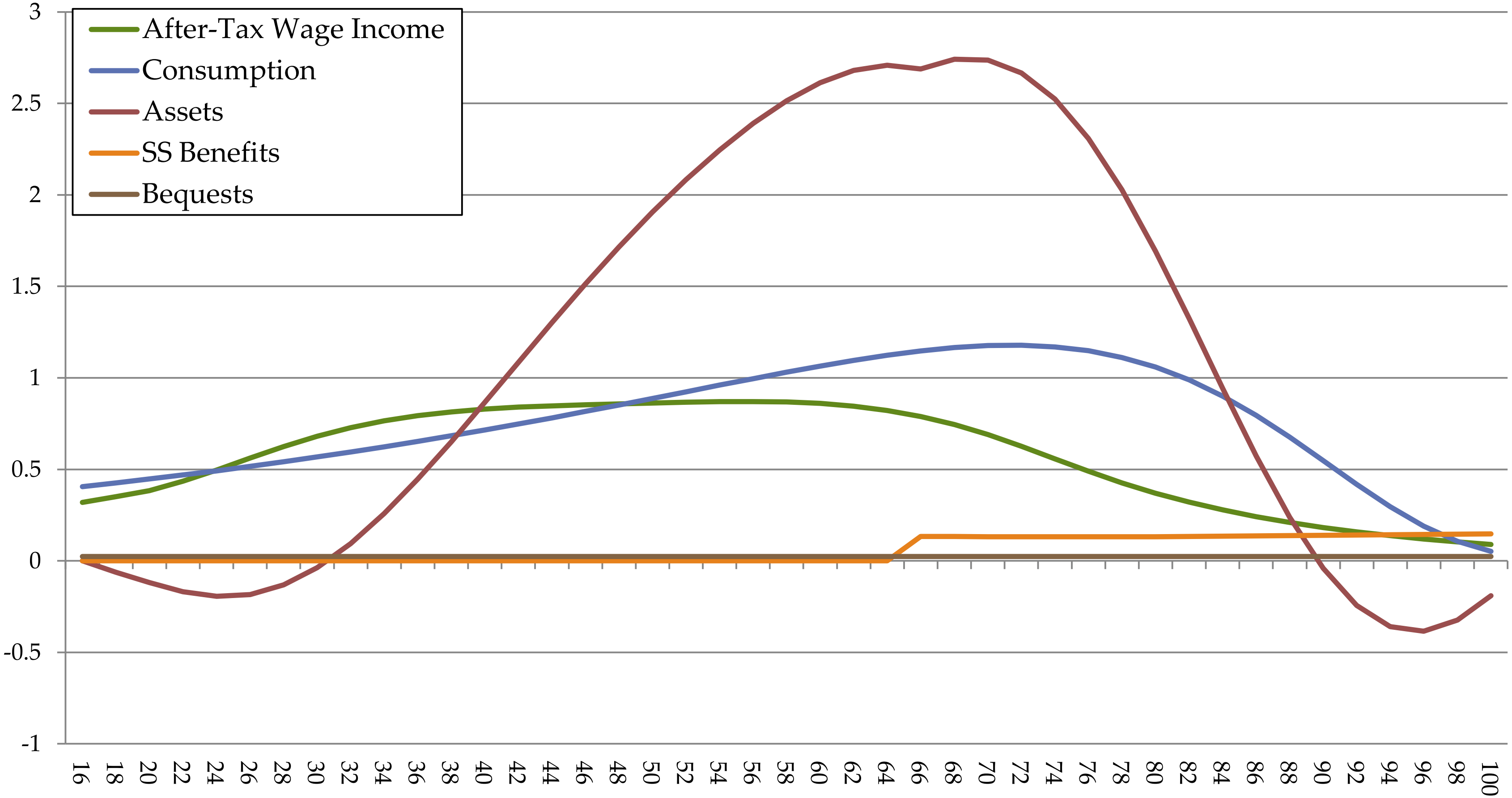
Steady State - Aggregate

α	0.33	$\bar{K}$	0.9379
γ	1	$\bar{H}$	0.0000
$\bar{g}^*$	0.005	$\bar{Y}$	0.8380
δ^*	0.1	$\bar{C}$	0.2784
β^*	0.98	$\bar{I}$	0.2180
θ	0.2	$\bar{L}$	0.7927
S	50	$\bar{r}^*$	0.1379
		$\bar{w}$	0.7082
ψ_z	.9	$\bar{T}$	0.0242
σ_z	.02	$\bar{B}$	0.0218
		$\bar{n}^*$	0.0094
		τ	0.0389

Steady State - Population



Steady State - Household



Linear Approximation - Example

Illustrate the technique first on a simple infintely-lived agent DSGE model.

Euler equation

$u^c\{c\} = \beta E\{u^c\{c'\}(1 + r' - \delta)\}$

Budget constraint

$c = w + (1 + r - \delta)k - k'$

Firm's optimization conditions

$r = \alpha K^{\alpha-1}(e^z)^{1-\alpha}Y$

$w = (1 - \alpha)K^{\alpha}(e^z)^{-\alpha}$

Technology

$$z' = Nz + e'; e' \sim iid(0, \sigma^2) \tag{5.1}$$

Linear Approximation - Example

First categorize variables into three categories:

1. Exogenous state variables (z)
2. Endogenous state variables (k)
3. Non-state variables (c, r, w)

Use a 1st-order Taylor-series expansion to linearize the log of the Euler equation. The other equations are used as definitions

$\beta E\{\, (c/c')^{\gamma}\ (1 + r' - \delta)\} = 1$

Write the approximation as:

$E\{F\tilde{k}'' + G\tilde{k}' + H\tilde{k} + L\tilde{z}' + M\tilde{z}\} = 0$

Where $\tilde{k}$ is the deviation of k from its steady state value

And $\tilde{z}$ is the deviation of z from its steady state value

Linear Approximation - Example

Assume the policy function for k' can be written as a linear approximation also.

$$\tilde{k}' = P\tilde{k} + Q\tilde{z}' \qquad (5.3)$$

Substitution of (5.3) & (5.1) into (5.2) yields

$$[(FP+G)P+H]\tilde{k} + [(FQ+L)N + (FP+G)Q + M]\tilde{z} = 0$$

Which requires P & Q to satisfy two conditions:

- $(FP+G)P+H=0$

  Involves solving a (matrix) quadratic in P.

- $(FQ+L)N+(FP+G)Q+M=0$

Linear Approximation – Our Model

Categorize variables into three categories:

1. Exogenous state variables (revealed now):
   $(\mathbf{Z}_t = z_t \,, \{\widehat{N}_t\}_{s=1}^{S})$
2. Endogenous state variables (chosen now):
   $(\mathbf{X}_t = \{\hat{k}_{s,t+1}\}_{s=E}^{S} \,, \{\hat{a}_{s,t+1}\}_{s=E+1}^{R-1}, \{\hat{b}_{s,t+1}\}_{s=R}^{S}, \widehat{H}_{t+1})$
3. Non-state variables (everything else)

There are S+1 exogenous state variables and 2(S-E)+1 endogenous state variables.

Linear Approximation – Our Model

We linearize a series of 2(*S*-*E*)+1 equations, using the model's equations as definitions as needed.

***S*-*E* Euler equations:**

$$\hat{c}_s^{\,-\gamma} = \beta E\{[\hat{c}_s'(1+g)]^{-\gamma}(1+r'-\delta)\} \qquad \text{for } 1 \le s \le S-1$$

***R*-*E*-1 AIME equations:**

$$\hat{a}_{s+1}'(1+g) = \tfrac{s-E-1}{s-E}\hat{a}_s + \tfrac{1}{s-E}\hat{w}\bar{\ell}_s \qquad \text{for } E \le s \le R-1$$

1 Initial benefits equation:

$$\hat{b}_R{}'(1+g) = \theta\left[\tfrac{R-2-E}{R-1-E}\hat{a}_{R-1} + \tfrac{1}{R-1-E}\hat{w}\bar{\ell}_{R-1}\right]$$

***S*-*R* later benefits equations:**

$$\hat{b}_{s+1}'(1+g) = \hat{b}_s \qquad \text{for } s > R$$

1 trust fund equation:

$$\hat{H}'(1+g) = \hat{H} + \textstyle\sum_{s=E}^{R-1}\hat{N}_s\tau\hat{w}\bar{\ell}_s - \sum_{s=R}^{S}\hat{N}_s\hat{b}_s$$

Linear Approximation – Our Model

Write this set of log linearized equations as

$$\mathrm{E}\{\mathbf{F}\widetilde{\mathbf{X}}'' + \mathbf{G}\widetilde{\mathbf{X}}' + \mathbf{H}\widetilde{\mathbf{X}} + \mathbf{L}\tilde{\mathbf{Z}}' + \mathbf{M}\tilde{\mathbf{Z}}\} = 0$$

We also have a set of S+1 linear equations that govern the motion of our exogenous state variables , which we write in matrix form as:

$$\tilde{\mathbf{Z}}' = \mathbf{N}\tilde{\mathbf{Z}} + \mathbf{e}$$

We can proceed as above and solve for the coefficients in the linearized policy function

$$\widetilde{\mathbf{X}}' = \mathbf{P}\widetilde{\mathbf{X}} + \mathbf{Q}\tilde{\mathbf{Z}}'$$

Baseline Simulation

We need to first calibrate an initial state.

For the population distribution we fit a polynomial to census data. Depicted in earlier figure.

We assume the technology level is one standard deviation below the mean, which is 98% of steady state productivity.

For capital stock and social security benefits across cohorts we assume the initial values are the same as the steady state.

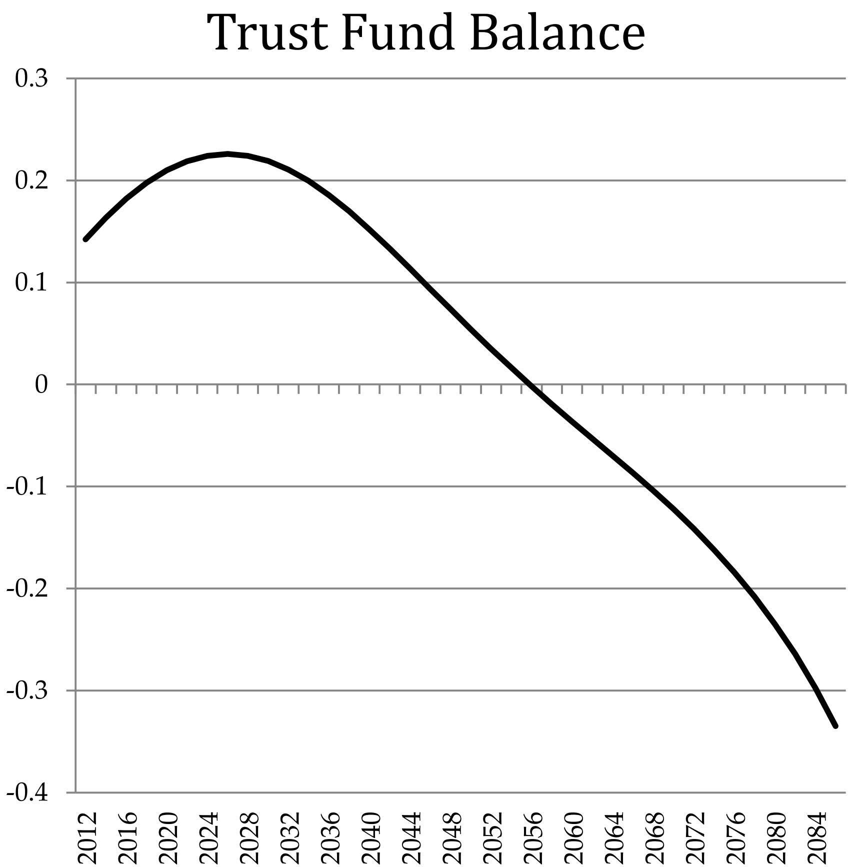
For AIME we assume the initial values are twice the steady state values.

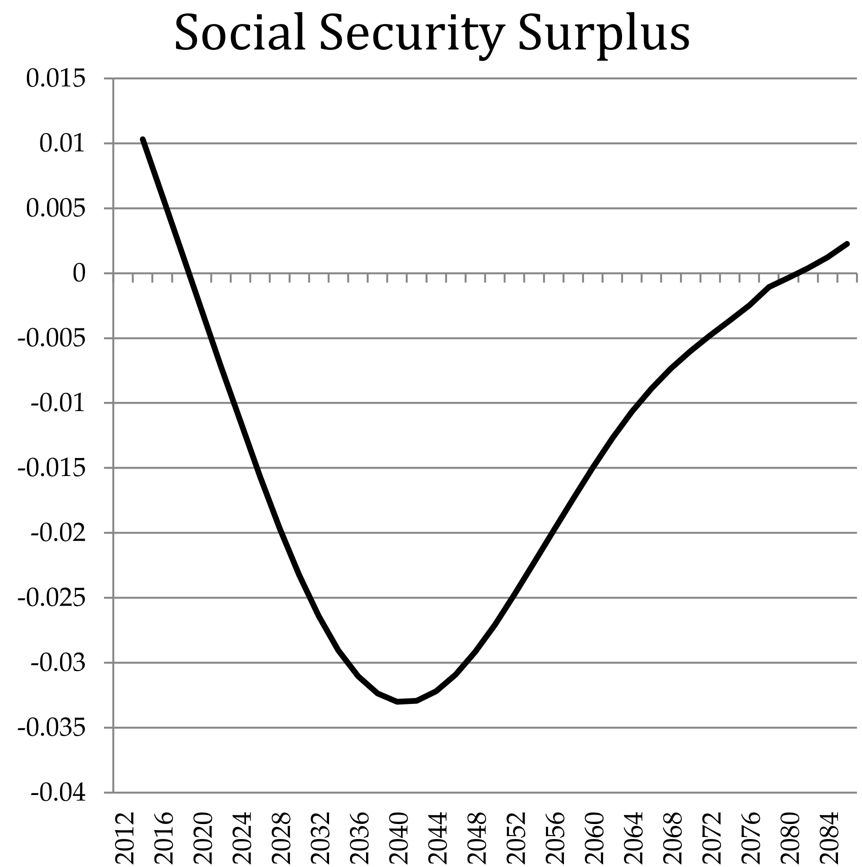
We assume the trust fund is initially 16.3% of GDP.

Simulate by imposing a series of zero shocks and let the model's dynamics move the economy back toward the steady state for a period of 75 years.

In our model, the trust fund has a unit root.

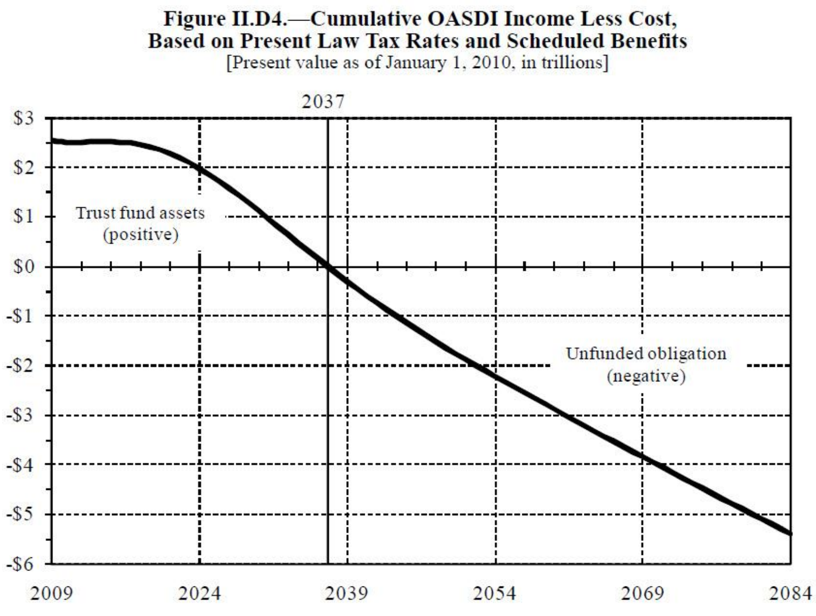
Baseline Simulation





Baseline Simulation

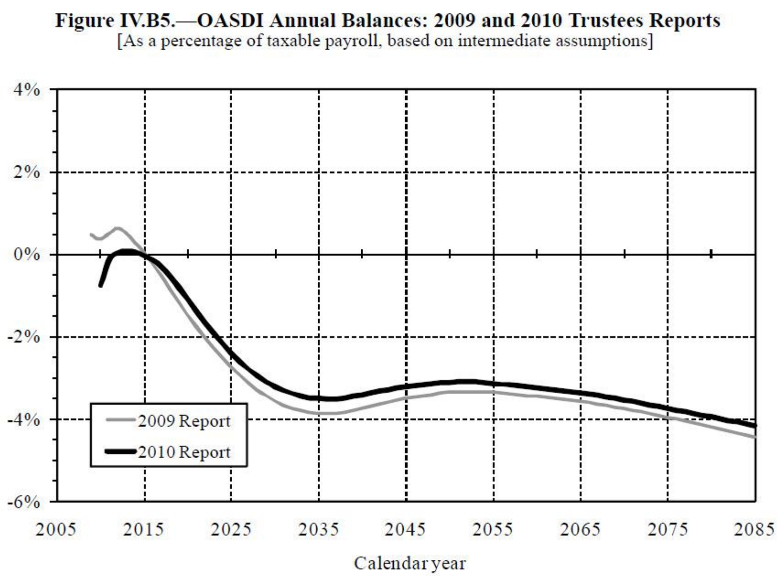
2010 Trustees Report



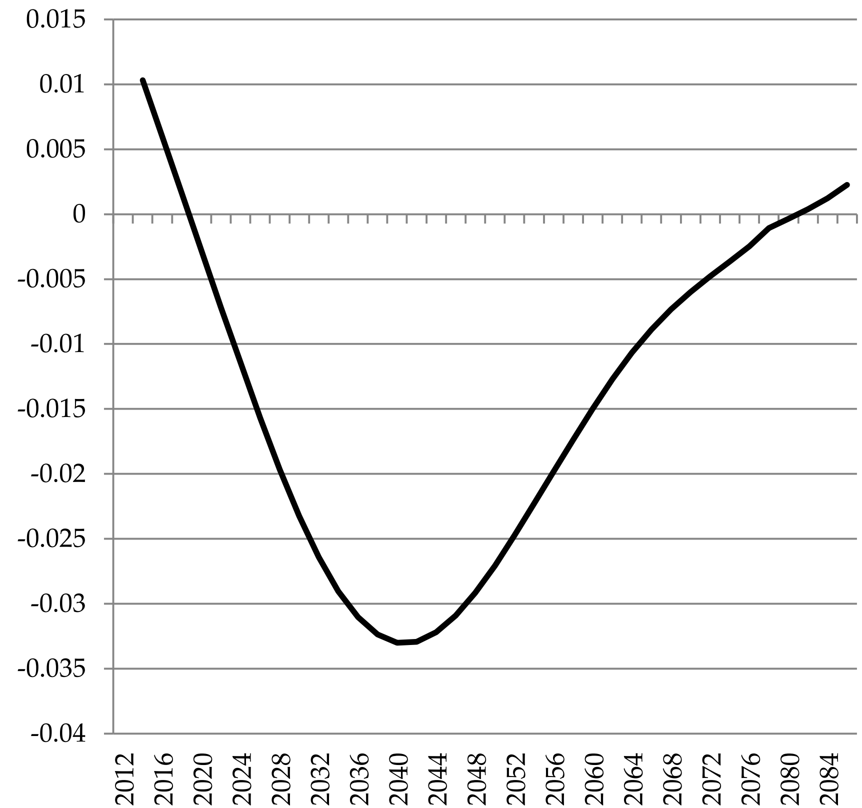
Our Model

Baseline Simulation

2010 Trustees Report



Our Model



Baseline Simulation

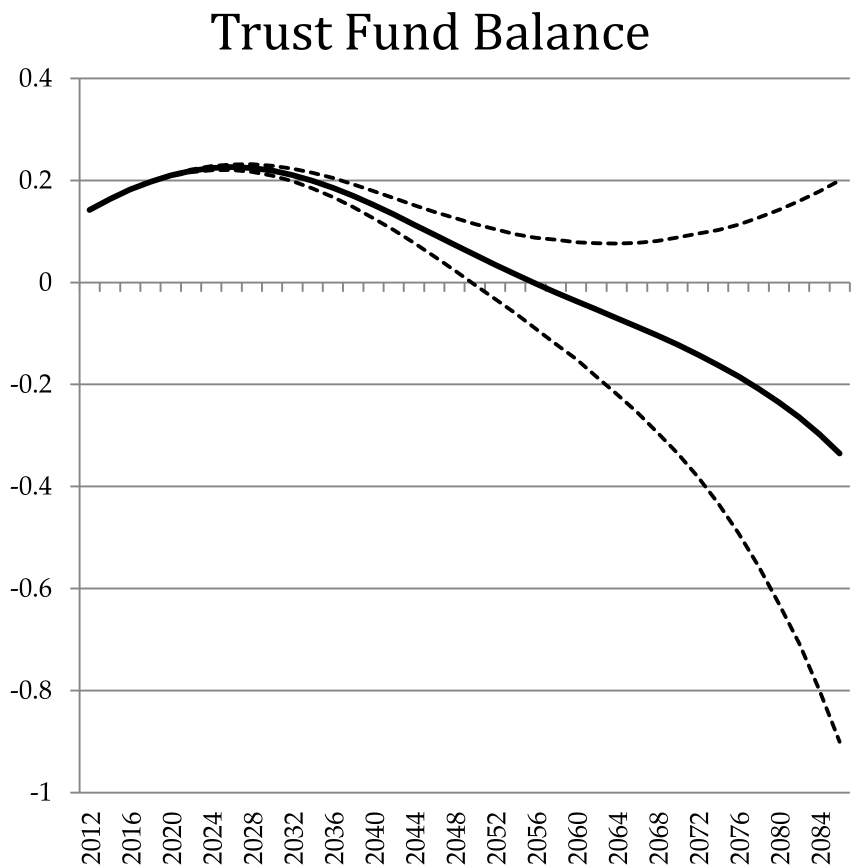
This simulation is a "best guess" scenario where all future shocks are assumed to be at their expected value of zero.

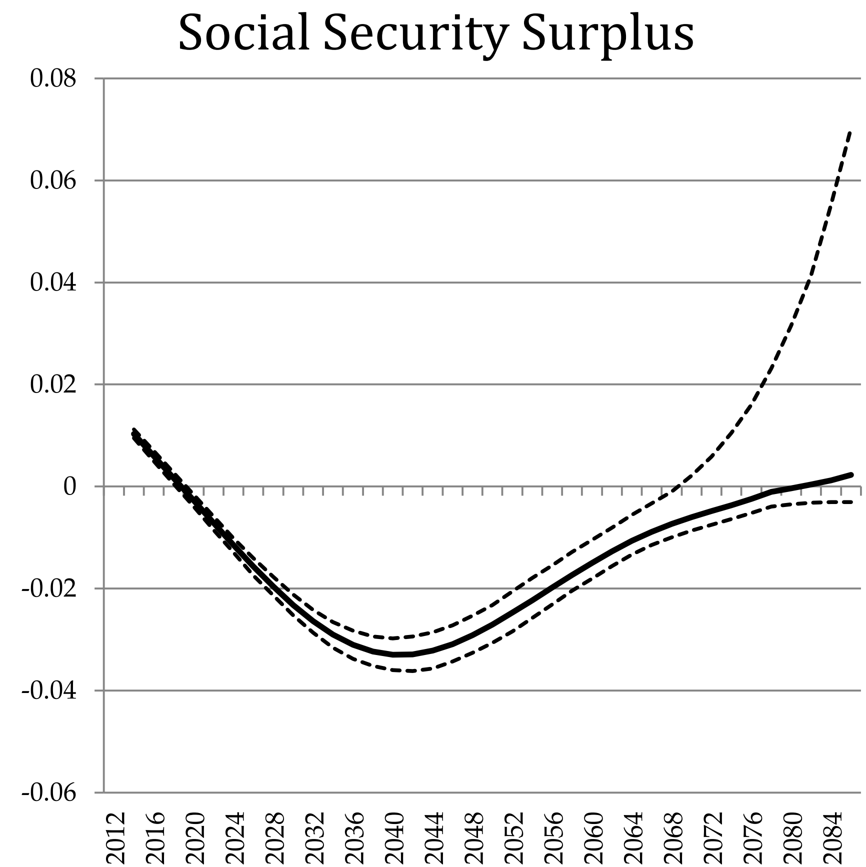
In reality, there will be stochastic shocks.

We impose zero variance on the demographic parameters, but allow a positive variance (.0004) and autocorrelation (.9 per annum) for the technology shocks.

We run 1000 Monte Carlos of 75 years each. We plot the 90% confidence bands around our original predictions from the previous slide.

Baseline Simulation





S=50, 1000 Monte Carlos

Stability Issues

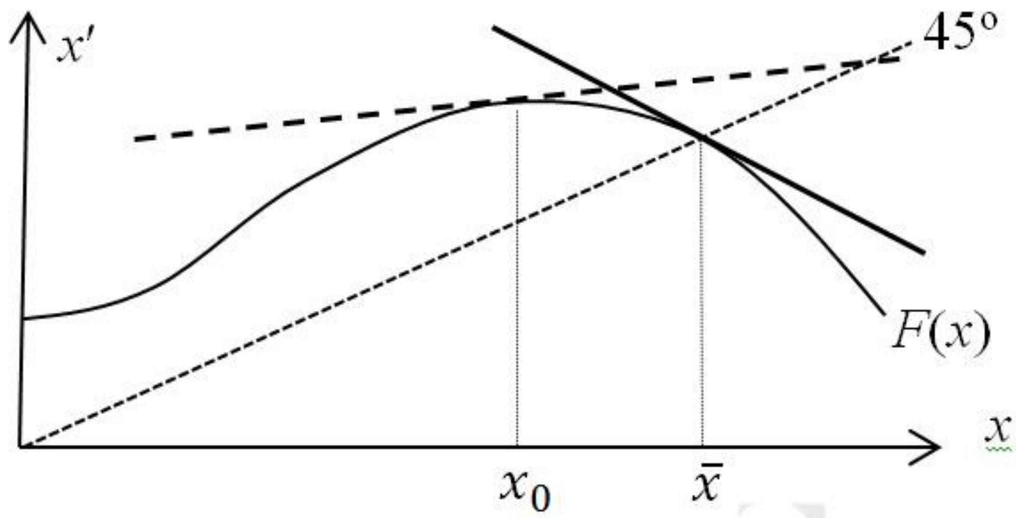
- Model is unstable.
- Higher balances on the trust fund earn greater interest payments, allowing the surpluses even when tax receipts exactly equal benefits payments
- Trust fund balances contribute to the total capital stock and will have influences on future wages and interest rates.

Linearizing about the Current State

- We need to be able to simulate a model that UNSTABLE or is NOT converging to a steady state.
- Change immigration ***holding benefits and tax parameters constant***, leading to unstable behavior for the trust fund.

- We could approximate our dynamic behavior equations about a point other than the steady state.

Linearizing about the Steady State



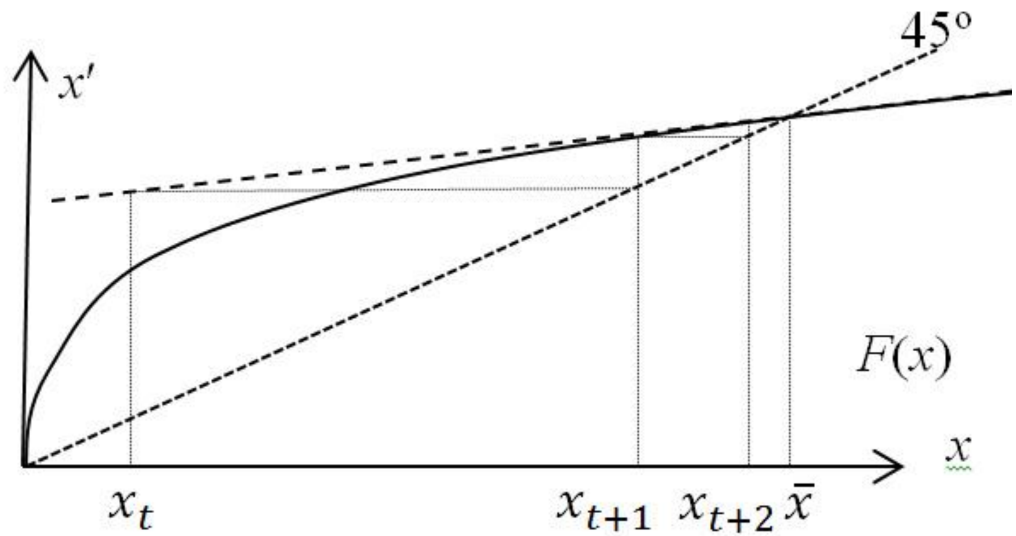
Accurate in the neighborhood of the steady state.

Less accurate the further away one gets from the steady state and the more nonlinear the true function is.

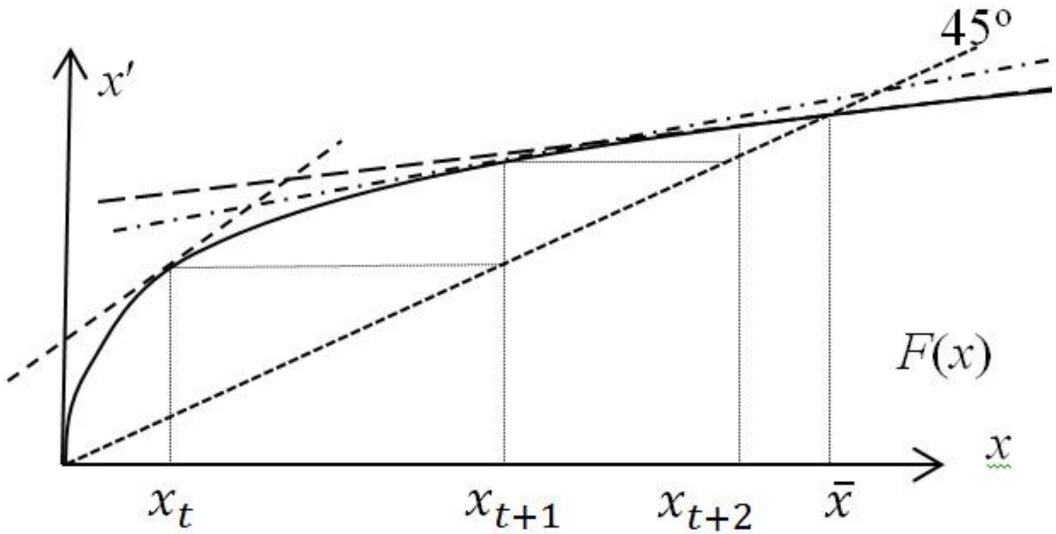
Linearized function may be very different if we choose a different point.

May also converge to a different steady state.

Linearizing about the Steady State



Convergence path to the when we use a function approximated about the steady state.



Linearizing about the Current State

Convergence path to the when we use a function approximated about the current state.

Requires approximating the function each period, rather than just once.

Linearizing about the Current State

Write the set of log linearized equations as

$$\mathrm{E}\{\mathbf{T} + \mathbf{F}\widetilde{\mathbf{X}}'' + \mathbf{G}\widetilde{\mathbf{X}}' + \mathbf{H}\widetilde{\mathbf{X}} + \mathbf{L}\tilde{\mathbf{Z}}' + \mathbf{M}\tilde{\mathbf{Z}}\} = 0$$

Where we are now considering deviation from the current values of X and Z, rather than the SS values.

Linear laws of motion:

$$\tilde{\mathbf{Z}}' = \mathbf{N}\tilde{\mathbf{Z}} + (\mathbf{N} - \mathbf{I})(\mathbf{Z}_0 - \bar{\mathbf{Z}}) + \mathbf{e}$$

Solve for the following linearized policy function:

$$\widetilde{\mathbf{X}}' = \mathbf{P}\widetilde{\mathbf{X}} + \mathbf{Q}\tilde{\mathbf{Z}}' + \mathbf{U}$$

Linearizing about the Current State

Iterative substitution yields:

$$[(\mathbf{FP}+\mathbf{G})\mathbf{P}+\mathbf{H}]\widetilde{\mathbf{X}}+[(\mathbf{FQ}+\mathbf{L})\mathbf{N}+(\mathbf{FP}+\mathbf{G})\mathbf{Q}+\mathbf{M}]\tilde{\mathbf{Z}}+\mathbf{T}+$$
$$[\boldsymbol{F}(\mathbf{I}+\mathbf{P})+\mathbf{G}]\mathbf{U}+(\mathbf{FQ}+\mathbf{L})(\mathbf{N}-\mathbf{I})(\mathbf{Z}_0-\bar{\mathbf{Z}})=\mathbf{0}$$

Which gives three conditions:

$$\mathbf{FP}^2+\mathbf{GP}+\mathbf{H}=\mathbf{0}$$

$$\mathbf{FQN}+(\mathbf{FP}+\mathbf{Q})\mathbf{G}+\mathbf{M}+\mathbf{LN}=\mathbf{0}$$

$$\mathbf{T}+[\boldsymbol{F}(\mathbf{I}+\mathbf{P})+\mathbf{G}]\mathbf{U}+(\mathbf{FQ}+\mathbf{L})(\mathbf{N}-\mathbf{I})(\mathbf{Z}_0-\bar{\mathbf{Z}})=\mathbf{0}$$

First two are same as before.

Last one gives:

$$\mathbf{U}=-[\boldsymbol{F}(\mathbf{I}+\mathbf{P})+\mathbf{G}]^{-1}[\mathbf{T}+(\mathbf{FQ}+\mathbf{L})(\mathbf{N}-\mathbf{I})(\mathbf{Z}_0-\bar{\mathbf{Z}})]$$

Linearizing about the Current State

Note that both $\widetilde{\mathbf{X}}$ and $\tilde{\mathbf{Z}}'$ will be zero if we are linearizing about the current state, $(\mathbf{X}, \mathbf{Z}')$, so that our linearized policy function becomes:

$$\widetilde{\mathbf{X}}' = \mathbf{U}$$

Hence, solving for $\mathbf{P}$ & $\mathbf{Q}$ is necessary only to obtain the correct value for $\mathbf{U}$.

$$\mathbf{U} = -[\boldsymbol{F}(\mathbf{I} + \mathbf{P}) + \mathbf{G}]^{-1}[\mathbf{T} + (\mathbf{FQ} + \mathbf{L})(\mathbf{N} - \mathbf{I})(\mathbf{Z}_0 - \bar{\mathbf{Z}})]$$

With high dimensionality of the state, solving for $\mathbf{P}$ & $\mathbf{Q}$ can be computationally burdensome.

Since we need to do this for each period this is a distinct disadvantage of this method.

Linearizing about the Current State

However, we could reduce computation time by implementing either of the following shortcuts:

- Shortcut 1 - Assume the policy function can be well-approximated by $\tilde{\mathbf{X}}' = \mathbf{U}$

  In this case there is no need to calculate $\mathbf{P}$ & $\mathbf{Q}$ and the formula for $\mathbf{U}$ is

  $\mathbf{U} = -[\boldsymbol{F} + \mathbf{G}]^{-\mathbf{1}}[(\mathbf{T} + \mathbf{L})(\mathbf{N} - \mathbf{I})(\mathbf{Z}_0 - \bar{\mathbf{Z}})]$

- Shortcut 2 - Use the steady state values of $\mathbf{P}$ & $\mathbf{Q}$ in the formula for $\mathbf{U}$.

  In this case we calculate $\mathbf{P}$ & $\mathbf{Q}$ only once about the steady state and then calculate $\mathbf{U}$ each period using these values as approximations of the values we would get if we were to linearize about the current state.

How well do these shortcuts work?

Linearizing about the Current State

MAD vs Exact Solution			Ratio to No Shortcut	
No Shortcut	Shortcut 1	Shortcut 2	Shortcut 1	Shortcut 2
Stochastic Fluctuations (250 observations, 1000 Monte Carlos)				
0.0243	0.0232	0.0232	0.9547	1.0000
0.0191	0.0199	0.0199	1.0419	1.0000
0.0122	0.0125	0.0125	1.0246	1.0000
0.0118	0.0107	0.0107	0.9068	1.0000
Smooth Convergence to Steady State (10 observations, 1 simulation)				
0.0593	0.0534	0.0529	0.9005	0.9906
0.0108	0.0089	0.0089	0.8241	1.0000
0.0056	0.0039	0.0039	0.6964	1.0000
0.0095	0.0051	0.0050	0.5368	0.9804
0.0105	0.0169	0.0143	1.6095	0.8462
Convergence to Steady State with Stochastic Shocks (250 observations, 1000 Monte Carlos)				
0.0144	0.0146	0.0145	1.0139	0.9932
0.0124	0.0127	0.0127	1.0242	1.0000
0.0123	0.0126	0.0126	1.0244	1.0000
0.0124	0.0127	0.0126	1.0242	0.9921

We run Monte Carlo experiments on a model where the exact solution is known.

- Log utility
- 100% depreciation
- Cobb-Douglas Production

We compare the mean absolute deviations (MAD)of these methods.

Linearizing about the Current State

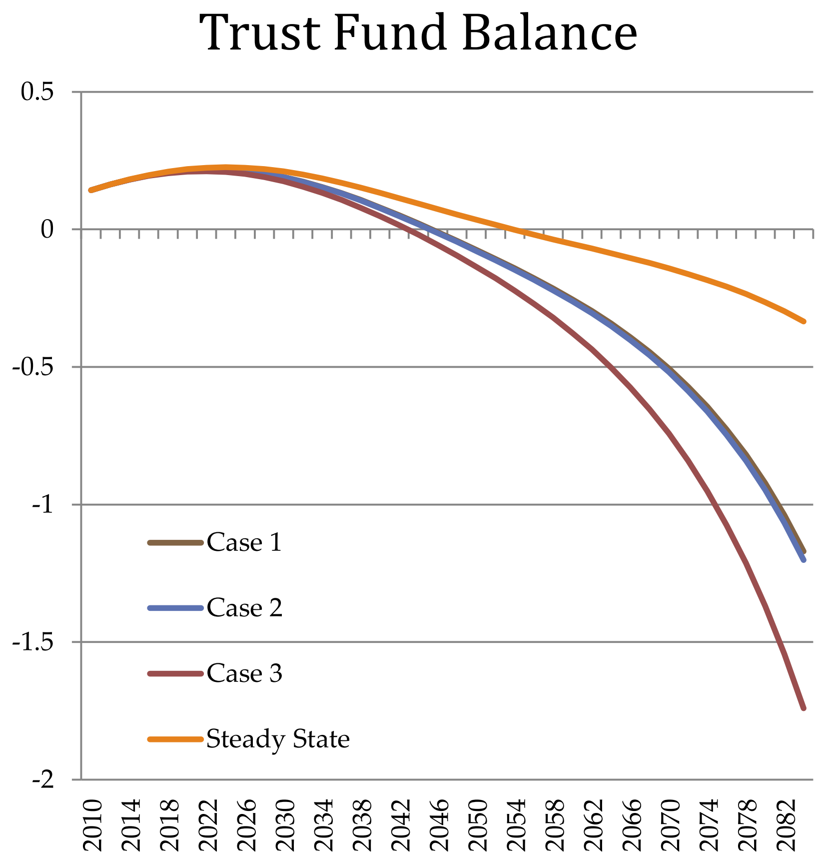
We run Monte Carlo experiments on this model where the exact solution is unknown.

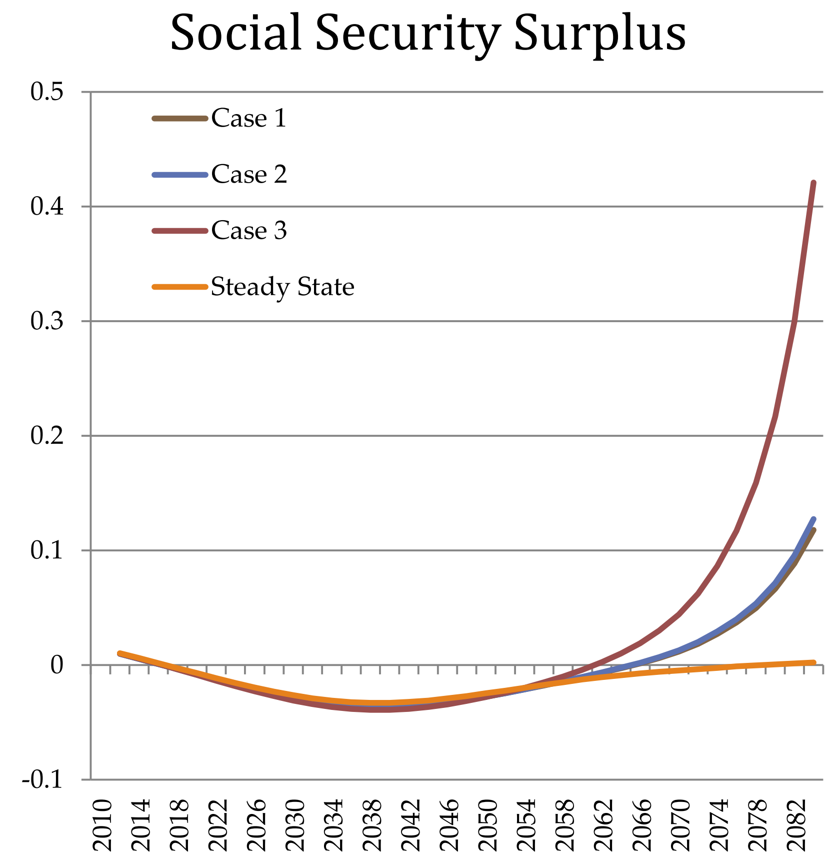
We compare the mean absolute deviations (MAD)of these methods.

Mean Absolute Deviation from Case 1 benchmark

	Case 2	Case 3	Steady State
H	0.0071	0.1228	0.1883
surplus	0.0010	0.0293	0.0141

Linearizing about the Current State





Linearizing about the Current State

We try simulating a version of our model with no steady state using the Shortcut 2 method.

Baseline Simulation

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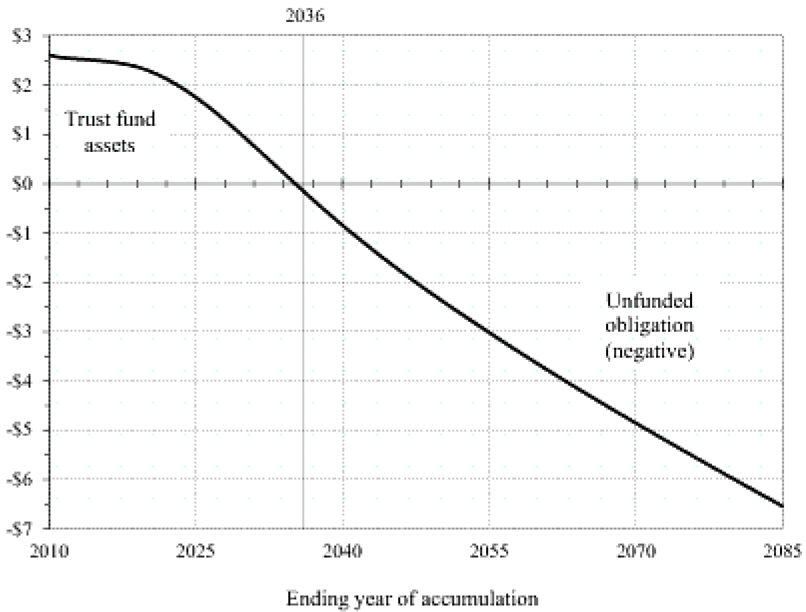
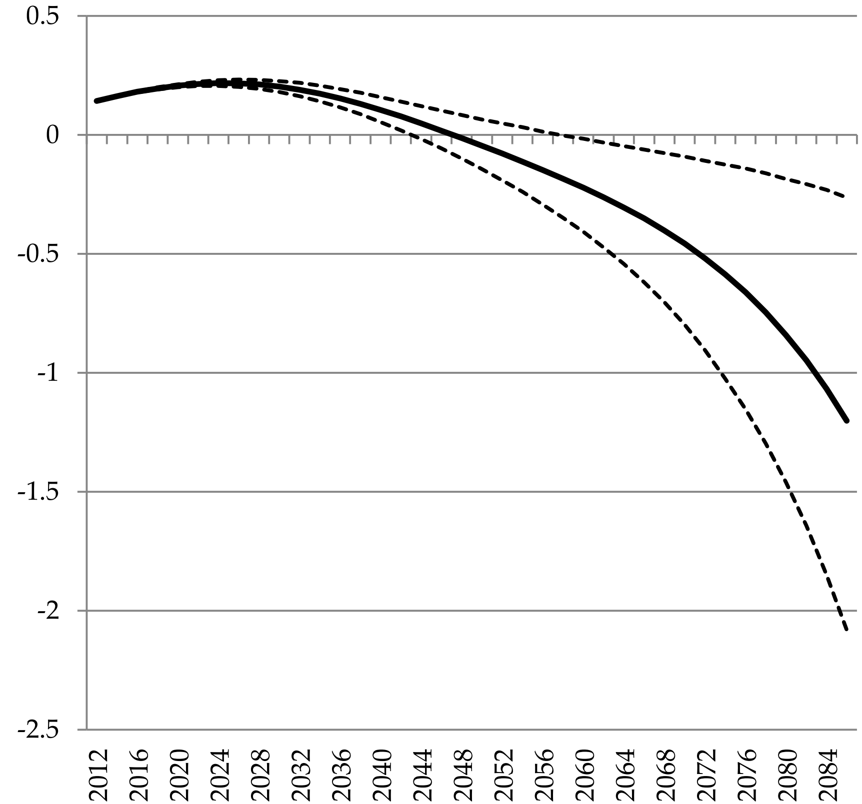


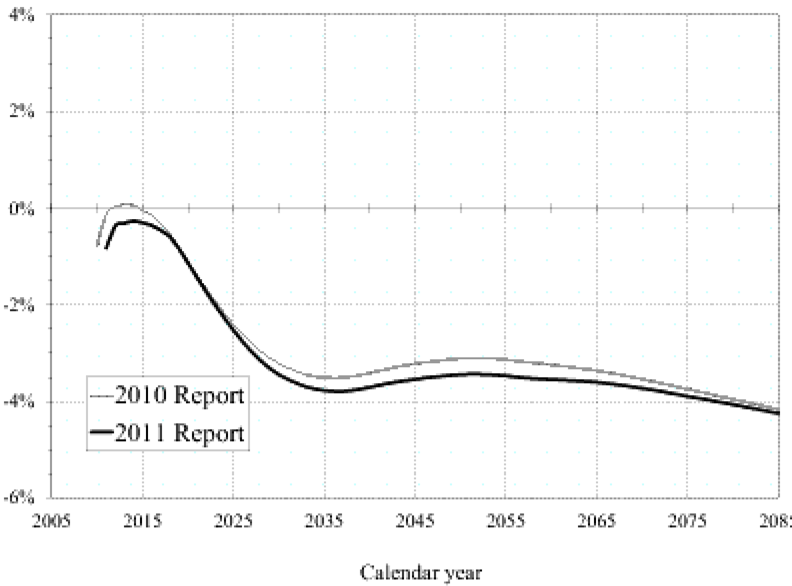
Our Model



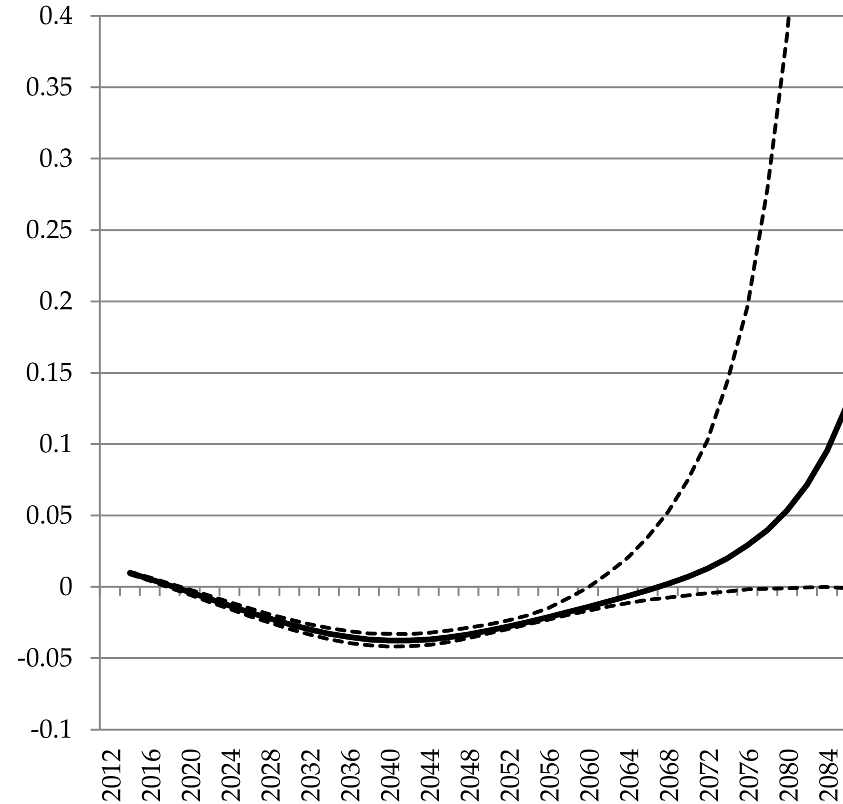
S=50, 1000 Monte Carlos

Baseline Simulation

2010 Trustees Report



Our Model



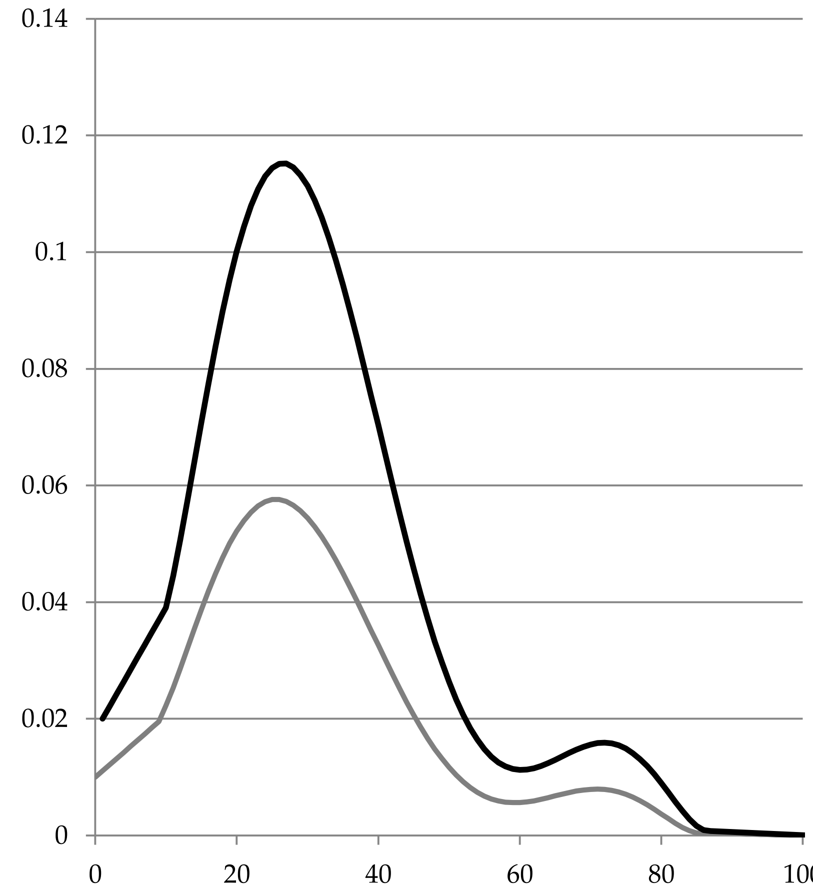
S=50, 1000 Monte Carlos

Baseline Simulation

	Trustees	Baseline
Surplus becomes negative	now	2020
Trust fund begins to fall	2022	2026
Trust fund falls below zero	2035	2048

Higher Immigration

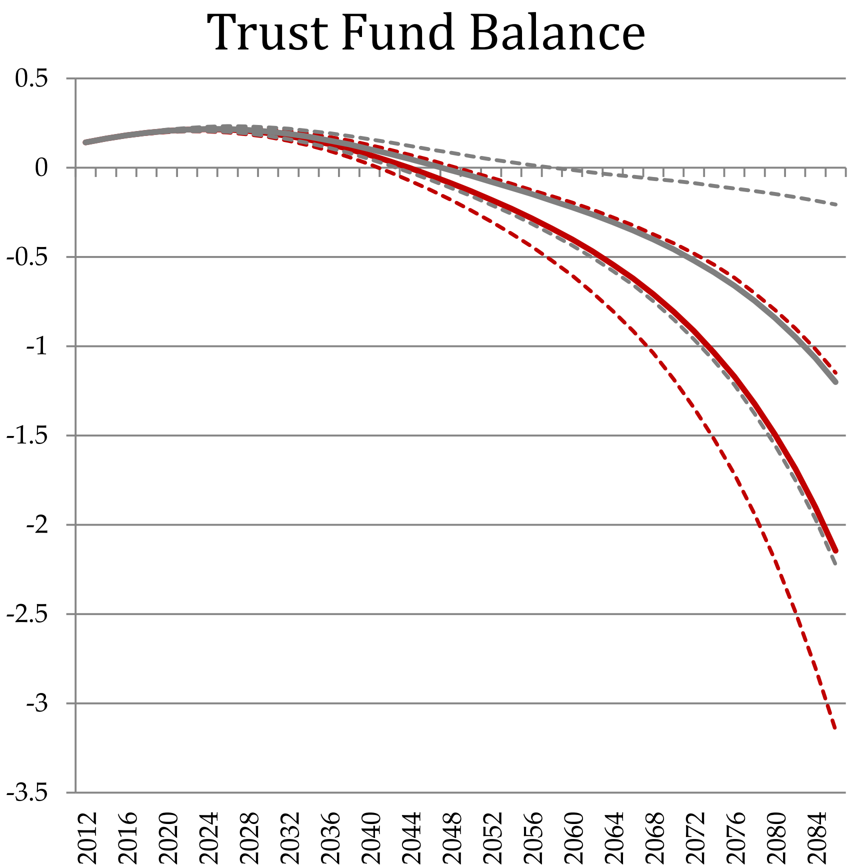
- Suppose we permanently doubled the immigration rates for each age cohort.

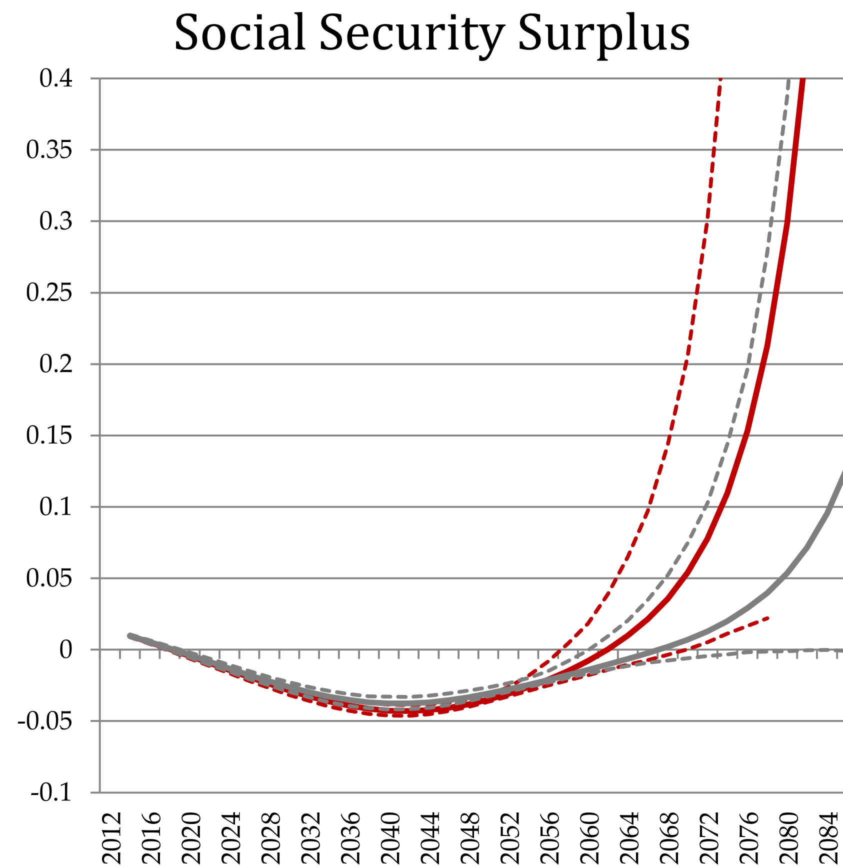


Higher Immigration

	Doubled ι	Baseline	Difference	Percentage
$\bar{K}$	0.9435	0.9379	0.0055	0.59%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8458	0.8380	0.0078	0.93%
$\bar{C}$	0.2777	0.2784	-0.0007	-0.24%
$\bar{I}$	0.2253	0.2180	0.0073	3.36%
$\bar{L}$	0.8015	0.7927	0.0087	1.10%
$\bar{r}^*$	0.1383	0.1379	0.0004	0.32%
$\bar{w}$	0.7071	0.7082	-0.0012	-0.17%
$\bar{T}$	0.0221	0.0242	-0.0021	-8.67%
$\bar{B}$	0.0206	0.0218	-0.0012	-5.63%
$\bar{n}^*$	0.0161	0.0094	0.0066	70.69%
τ	0.0363	0.0389	-0.0025	-6.51%

Higher Immigration





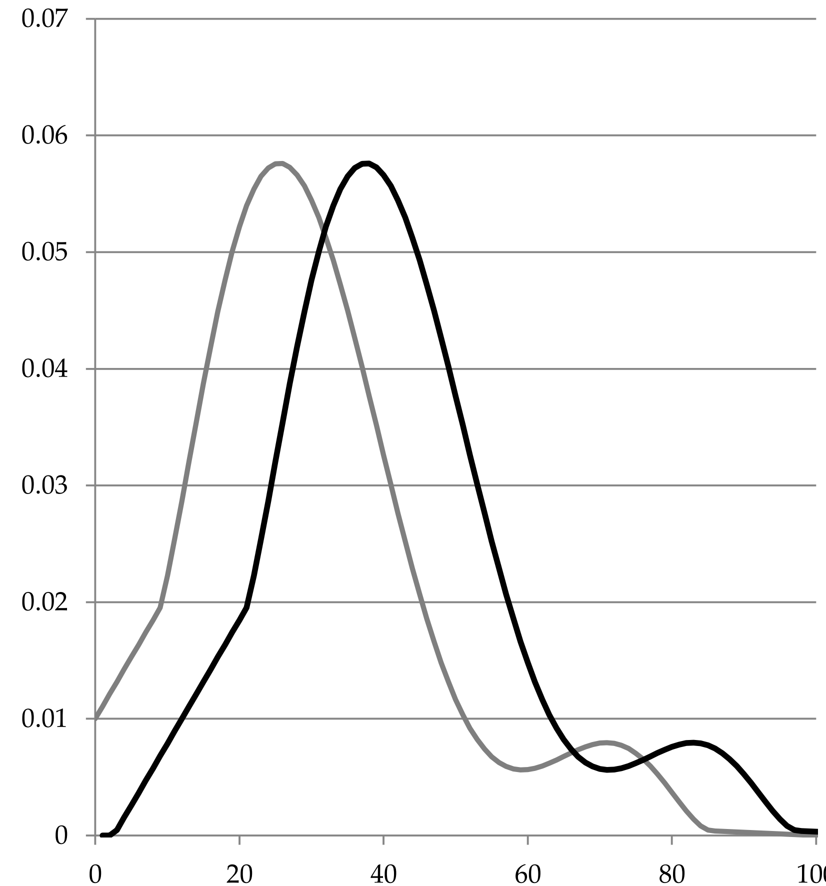
S=50, 1000 Monte Carlos

Higher Immigration

	Trustees	Baseline	Doubled Immigration
Surplus becomes negative	now	2020	2020
Trust fund begins to fall	2022	2026	2026
Trust fund falls below zero	2035	2048	2044

Skewed Immigration

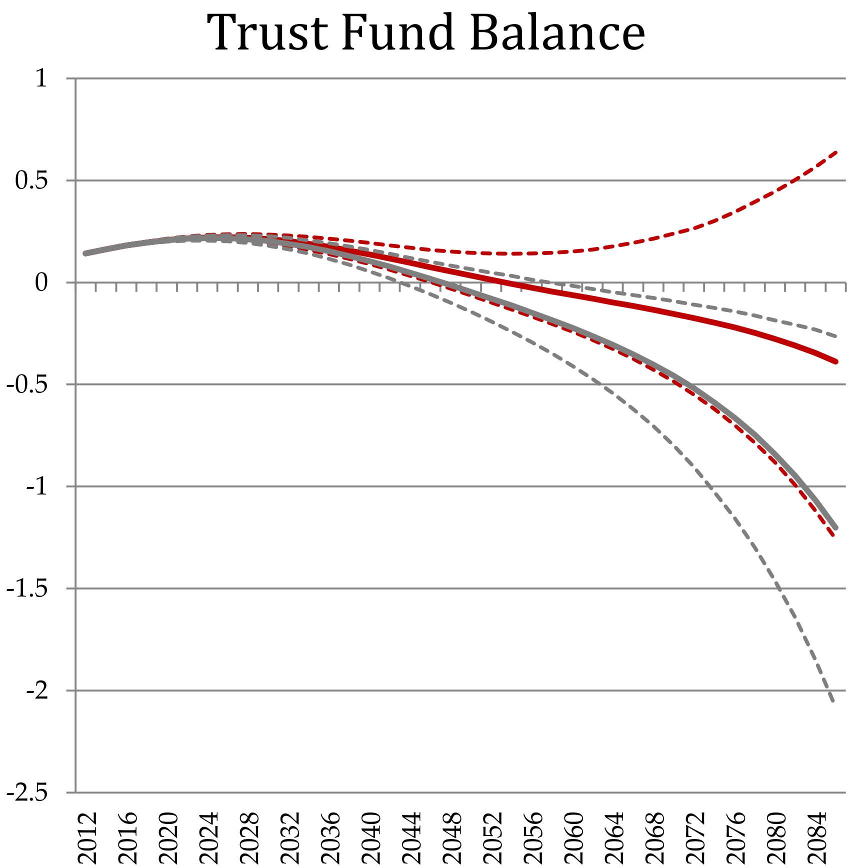
- Suppose we encouraged the immigration of older immigrants

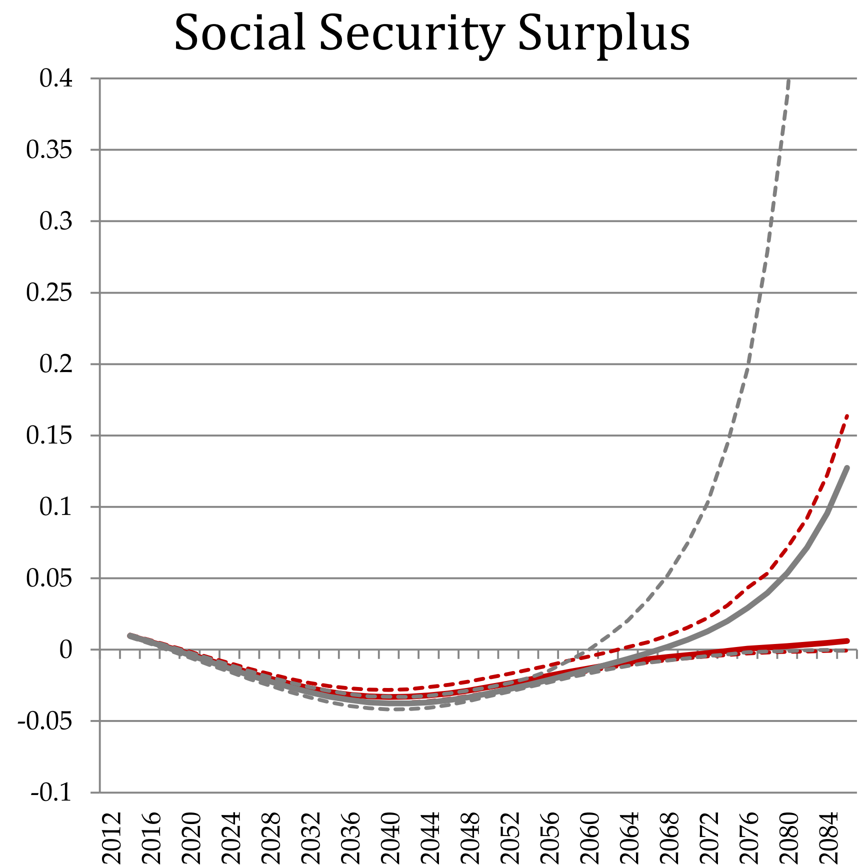


Skewed Immigration

	Skewed	Baseline	Difference	Percentage
$\bar{K}$	0.9608	0.9379	0.0229	2.44%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8542	0.8380	0.0162	1.94%
$\bar{C}$	0.2865	0.2784	0.0081	2.91%
$\bar{I}$	0.2154	0.2180	-0.0026	-1.20%
$\bar{L}$	0.8061	0.7927	0.0134	1.69%
$\bar{r}^*$	0.1373	0.1379	-0.0006	-0.46%
$\bar{w}$	0.7100	0.7082	0.0017	0.24%
$\bar{T}$	0.0263	0.0242	0.0021	8.62%
$\bar{B}$	0.0243	0.0218	0.0025	11.34%
$\bar{n}^*$	0.0065	0.0094	-0.0029	-31.09%
τ	0.0424	0.0389	0.0036	9.22%

Skewed Immigration





S=50, 1000 Monte Carlos

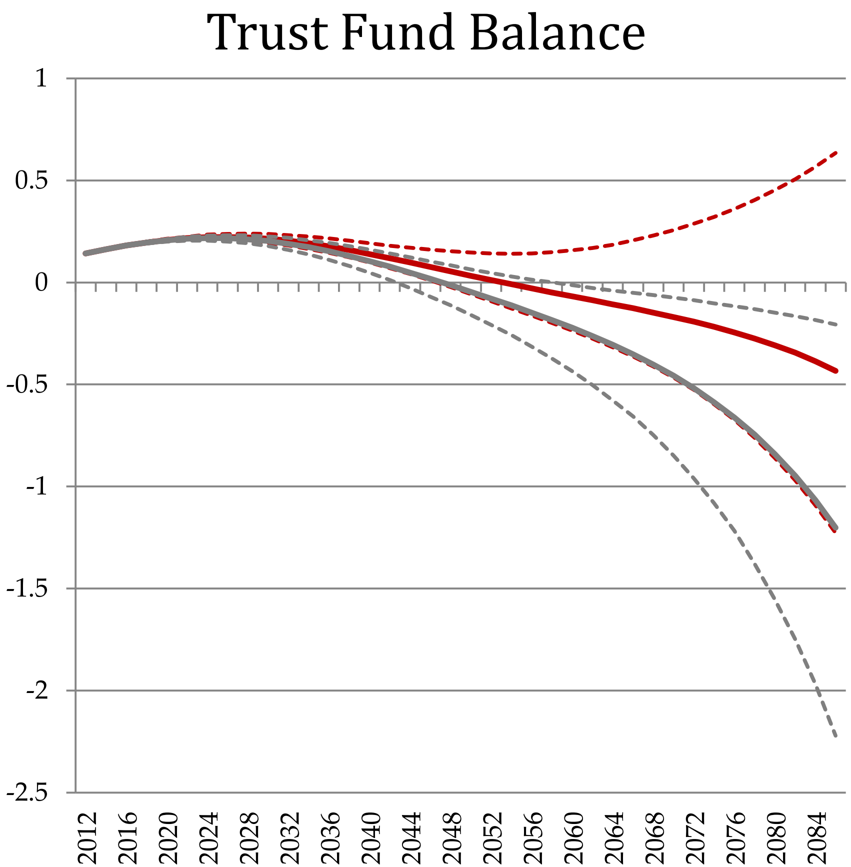
Skewed Immigration

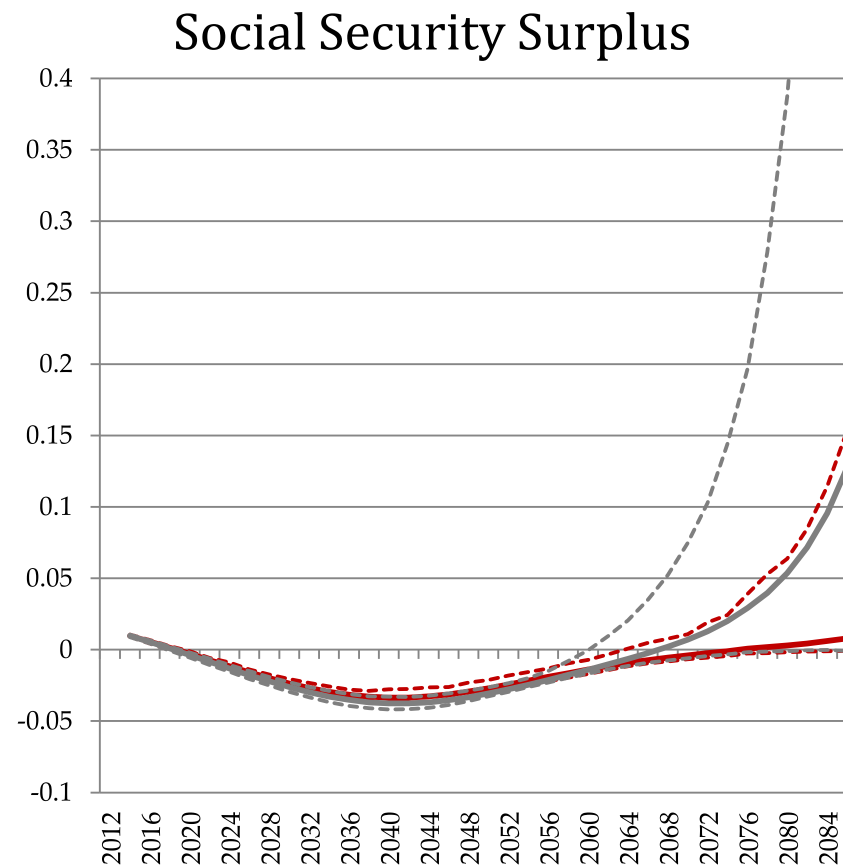
	Trustees	Baseline	Skewed Immigration
Surplus becomes negative	now	2020	2020
Trust fund begins to fall	2022	2026	2028
Trust fund falls below zero	2035	2048	2054

Doubled & Skewed Immigration

	Doubled ι	Baseline	Difference	Percentage
$\bar{K}$	0.9885	0.9379	0.0505	5.39%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8777	0.8380	0.0397	4.74%
$\bar{C}$	0.2935	0.2784	0.0151	5.42%
$\bar{I}$	0.2204	0.2180	0.0024	1.08%
$\bar{L}$	0.8278	0.7927	0.0351	4.43%
$\bar{r}^*$	0.1371	0.1379	-0.0008	-0.57%
$\bar{w}$	0.7104	0.7082	0.0021	0.30%
$\bar{T}$	0.0260	0.0242	0.0018	7.59%
$\bar{B}$	0.0254	0.0218	0.0036	16.46%
$\bar{n}^*$	0.0103	0.0094	0.0009	9.17%
τ	0.0432	0.0389	0.0043	11.18%

Doubled & Skewed Immigration



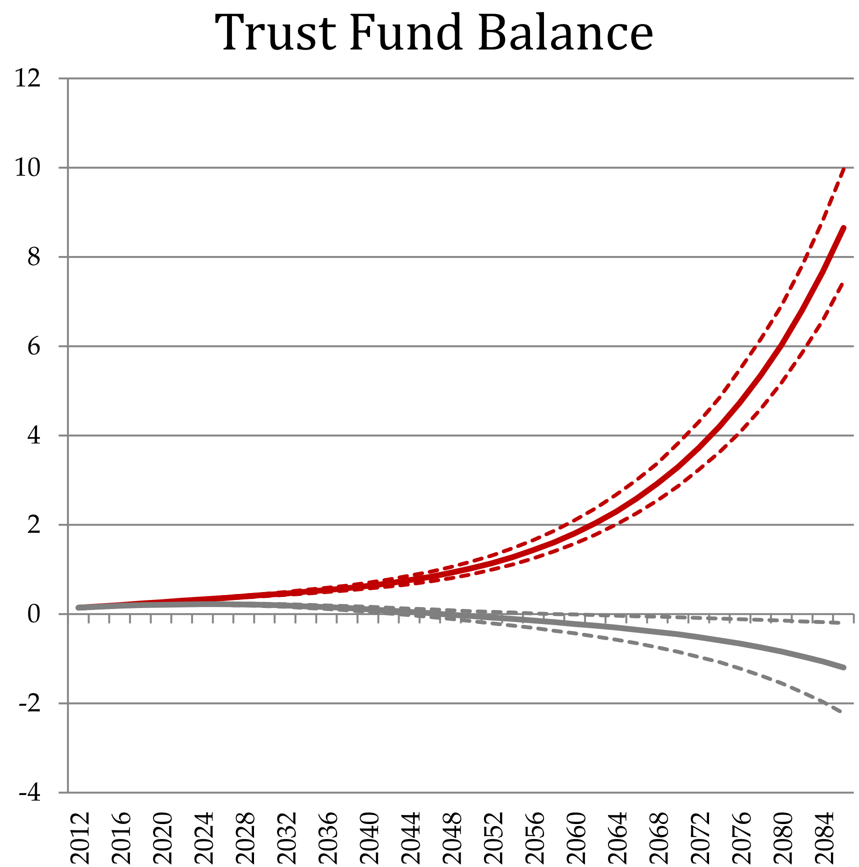


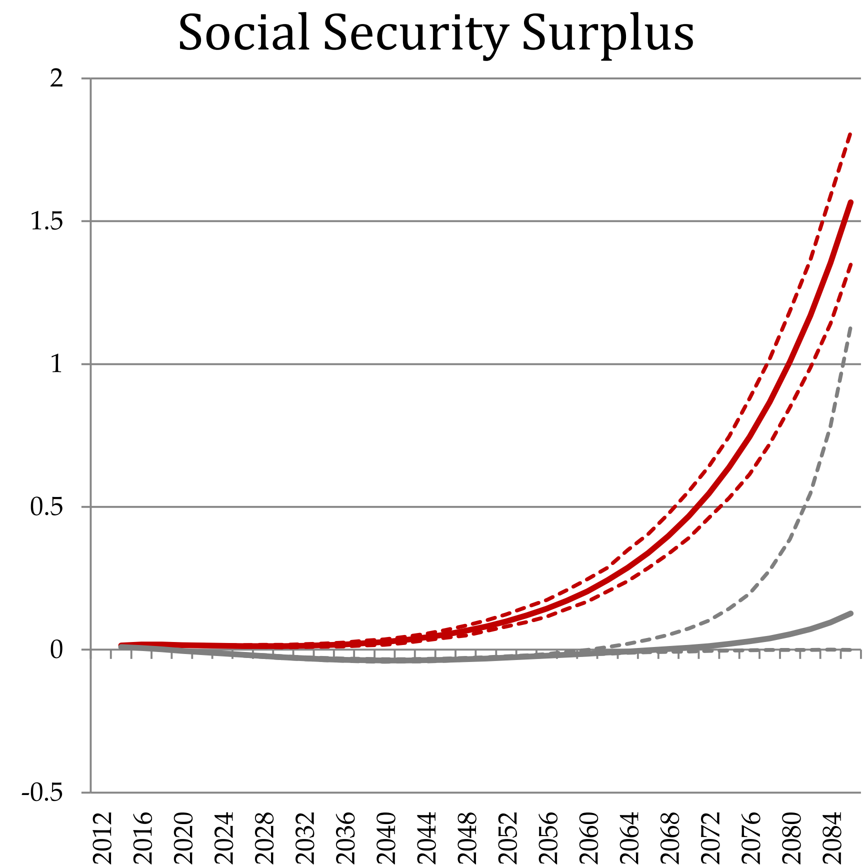
S=50, 100 Monte Carlos

Doubled & Skewed Immigration

	Trustees	Baseline	Doubled Immigration	Skewed Immigration
Surplus becomes negative	now	2020	2020	2020
Trust fund begins to fall	2022	2026	2026	2028
Trust fund falls below zero	2035	2048	2044	2054

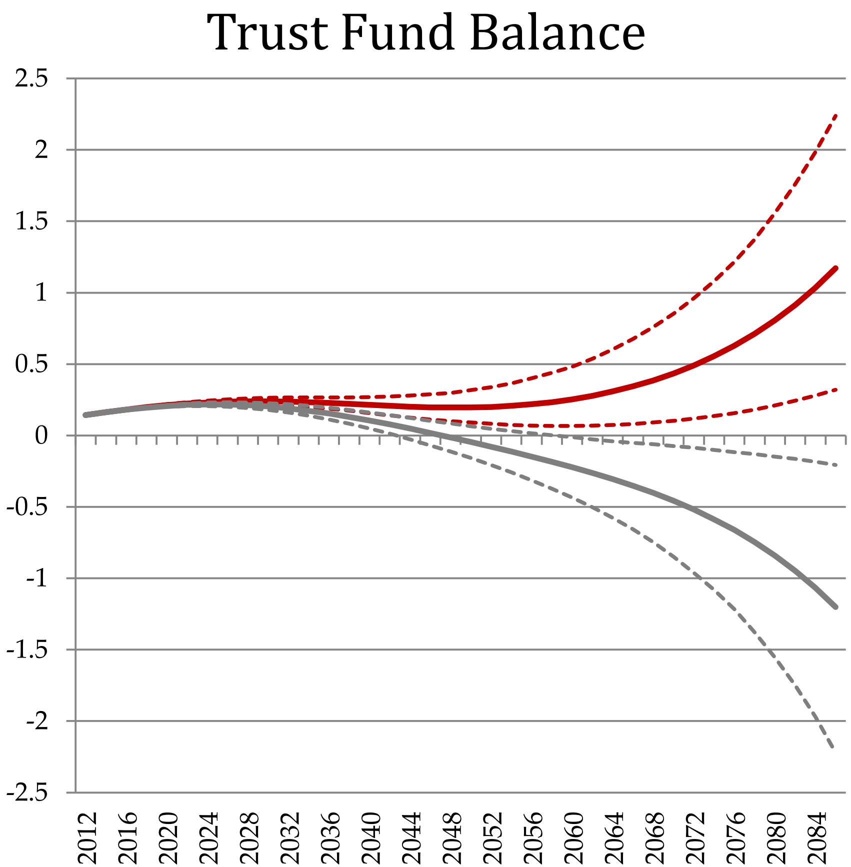
Retirement Age of 70

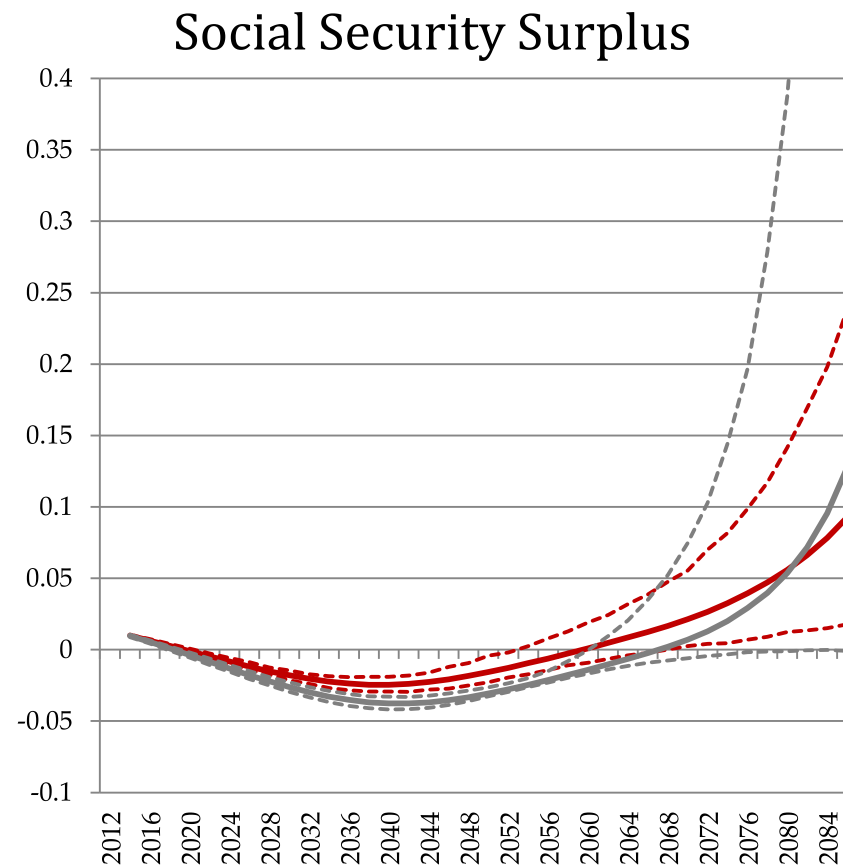




S=50, 100 Monte Carlos

10% Lower Benefits



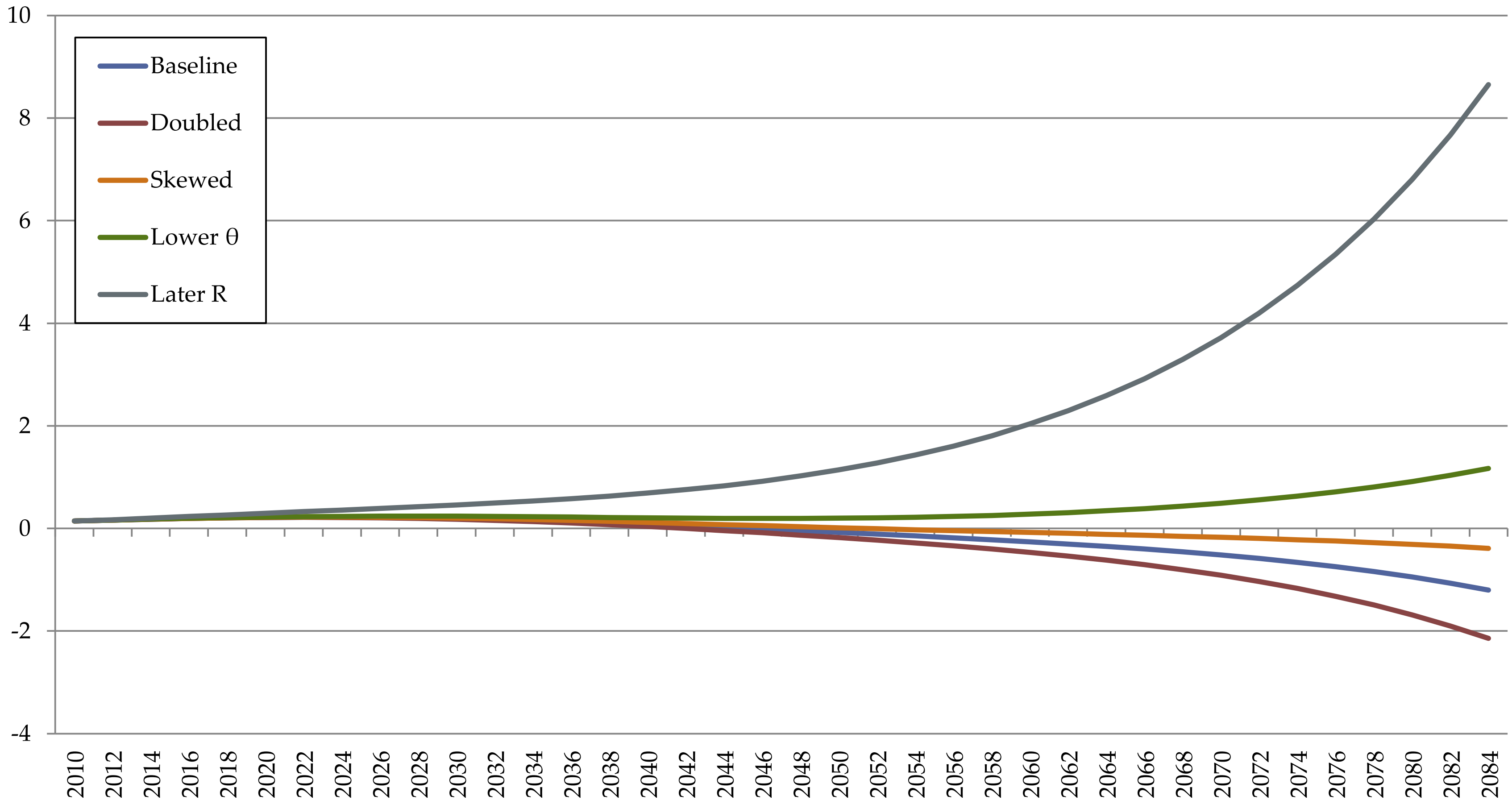


S=50, 100 Monte Carlos

Lower Benefits

	Trustees	Baseline	Later Retirement	Lower Benefits
Surplus becomes negative	now	2020	never	2020
Surplus becomes positive again	never	never	n/a	2060
Trust fund begins to fall	2022	2026	never	2030
Trust fund rises again	never	never	n/a	2048
Trust fund falls below zero	2035	2048	never	never

Comparison of Trust Fund Balances



Linearizing about Cohort Averages

In this case we have a set of cohort averages of steady state savings levels, $\{\bar{b}_s\}$, and log-linearize all the Euler equations for individuals of the same age about these values. This gives set of equations:

$\breve{\boldsymbol{\Gamma}} = \beta E\left\{\mathbf{T} + \mathbf{F}\breve{\mathbf{X}}_{t+1} + \mathbf{G}\breve{\mathbf{X}}_t + \mathbf{H}\breve{\mathbf{X}}_{t-1} + \mathbf{L}\widetilde{\mathbf{Z}}_{t+1} + \mathbf{M}\widetilde{\mathbf{Z}}_t\right\} = \mathbf{0}$, where $\breve{\mathbf{X}}_t$ is a vector of deviations of the X's from the cohort averages.

The policy functions are assumed to take the following form:

$\breve{k}_{i,s+1,t+1} = P_{Ks}\widetilde{K}_t + P_{ks}\breve{k}_{i,s,t} + \mathbf{Q}_s\tilde{\mathbf{z}}_t + \mathbf{U}_{i,s,t}$

Linearizing about Cohort Averages

One could also take averages over subsets of a cohort. For example, the averages over individuals with the same ability over past ability histories. One would then log-linearize about these average steady state values, $\{\bar{k}_{s,i}\}$.

The dimensionality of this problem is related to $(S-1)IH$ kinds of individuals.

Linearizing about Cohort Averages

As a practical matter simulating with $(S-1)IH$ individuals can be intractable even if the decision rules are found using approximations with smaller dimensionality.

One way to solve this is to discretize the allowable values of savings for individuals.

In this case the past history would be completely summarized by the current value of savings and individuals with the same level of savings would be identical regardless of the ability history that led them to that level of savings.

This greatly reduces the dimensionality. We have already implemented this methodology in our earlier paper.