When the Social Security Trust Fund Runs Out: Linearization about the Current State

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Abstract

We estimate the time until the Social Security trust fund runs out by simulating an overlapping generations model with stochastic life spans, immigration, aggregate shocks, and a tax and transfer policy calibrated to the U.S. economy. This class of fiscal policy problems also highlights the need for a solution method that can accommodate unstable steady states and nonstationarity. We detail such a solution method in which we linearize the model around the current state each period, updating the approximated characterizing equations each period. Our simulations imply that the Social Security trust fund is likely to run out in 38 years. However, 95 percent confidence intervals suggest that the trust fund could run out anytime between 34 and 50 years from now.

1. Introduction and Literature Review

A large amount of current research is focused on the effects of changing fiscal policy in the United States, both with regard to countercyclical policy (see Christiano, et al (2010), Kumhof, et al (2010), and Zubiary (2010)) and to reducing the national debt (see Gomes, et al (2010) and Traum and Yang (2010)). Regarding questions about reducing the national debt, the two main contributors to U.S. deficit spending now and long into the future are the Social Security system and the government health care benefits of Medicare and Medicaid (see CBO (2010)). Because Social Security policy (as well as Medicare and Medicaid policy) affect age cohorts differently, overlapping generations (OLG) dynamic stochastic general equilibrium (DSGE) models are the theoretical tool of choice for these studies.

In this paper, we calibrate an OLG model with stochastic life spans, immigration, aggregate shocks, and a tax and transfer program similar to Social Security to the United States. We simulate this model in order to estimate the time until the Social Security trust fund runs out, as well as 95 percent confidence intervals around that point estimate. Our simulations imply that the Social Security trust fund is likely to run out in ? years. However, 95 percent confidence intervals suggest that the trust fund could run out anytime between ? and ? years from now.

An additional contribution of this paper is that we detail a solution method for the broad class of DSGE models that have unstable steady states and are characterized by nonstationarity. Recent official projections have noted that the current state of U.S. tax policy is not sustainable (see CBO (2010) and GAO (2007)) and, therefore, is not a steady-state. However, current DSGE solution methods rely on the models exhibiting long-run stationarity. Our solution method accommodates nonstationarity by linearizing the characterizing equations of the model around the current state each period and updating those approximations each successive period. This solution technique is similar to the iterated unscented Kalman filter used in engineering and operations research (see Banani and Masnadi-Shirazi (2007)) but does not have the same latent variable assumptions as the Kalman filter.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents a stationary version of the model. Section 4 presents the unstable steady state and the calibration. Section 5 details the updating linearization around the current state solution method and a simulation of the trust fund. Section 6 presents how some policy experiments change the simulated time path of the trust fund balance, and Section 7 concludes.

2. The Model

Demographics

Households live for a maximum of S periods. Each period a new cohort of households is born and some portion of existing households of all ages die. In addition, each period new households of various ages immigrate into the economy. The populations of households of various ages evolve according to the following laws of motion.

$$N'_{s+1} = N_s(\rho_{s+1} + \iota_{s+1}) \text{ for } 1 \leq s \leq S-1 \tag{2.1}$$

Where N_s is the population aged s, ρ_{s+1} is the probability the household lives to age $s+1$ given it has already lived to age s, ι_{s+1} is the immigration rate for households as a faction of the current age s population. ρ_{s+1} and ι_{s+1} could be stochastic. A prime on a variable (′) denotes its value in the following period.

Age 1 households arrive via birth after all immigration has occurred and agents are one period older.

$$N'_1 = \sum_{s=1}^{S} f_s N_s \tag{2.2}$$

Where f_s is the fertility rate for households of age s, and could also be stochastic.

Households

The objective of existing households is to maximize the expected value of utility over their lifetime. All households are endowed with the same amount of labor at a given age. We assume they do not work when young, prior to age E, and cease working at an exogenously given retirement age of R.

Households accumulate capital over time by saving a portion of their wage income. They also receive a transfer payment (denoted T) each period which are the proceeds from liquidating the capital of the households that die at the end of the previous period. Finally,

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households participate in a public social security system by paying a portion of their wage income in taxes up to age $R - 1$, and receiving a benefit payment (denoted b) each period thereafter until death.

For ease of analysis we choose to set up the households' problems as dynamic programs and write them using Bellman equations. For individuals in a generic cohort, aged s this is:

$$V_s(\Omega) = \max_{k_{s+1}'} u\{c_s\} + \beta \rho_{s+1} E\{V_{s+1}(\Omega')\}$$

Where Ω is the information set, $u\{.\}$ is the within-period utility function, and β is the household's subjective discount factor. Note that because households do not live forever, their value functions vary by age.

Household consumption is defined by the following budget constraint.

$$c_s = w\bar{\ell}_s(1 - \tau) + (1 + r - \delta)k_s - k_{s+1}' + b_s + T \tag{2.3}$$

for $1 \leq s \leq S$

Where w is the wage rate, $\bar{\ell}_s$ is the household endowment of labor ate age s, τ is the tax rate on labor income, r is the return on capital, δ is the rate of capital depreciation, k_s is the household's holdings of bonds coming into the period, b_s is the pension benefit payment received, and T is a lump-sum transfer.

The solution gives the following Euler equation:

$$u^c\{c_s\} = \beta \rho_{s+1} E\{u^c\{c_s'\}(1 + r' - \delta)\} \tag{2.4}$$

Where $u^c\{.\}$ denotes the marginal utility of consumption.

We use versions of equation (2.4) for $1 \leq s \leq S - 1$.

In order to solve for its own transition function, $k_{s+1}' = k_s(\Omega)$, the household needs know the value functions for ages s and s+1 and it needs to form an expectation of the aggregate capital stock, K'. This means it also needs to know the transition functions of all the other households and their arguments. The transition functions for the oldest cohort are trivial. Since $V_{S+1}(\Omega') = 0$, the household will choose $k_{S+1}' = 0$. Transition functions for other cohorts will be found using numerical techniques explained below.

Firms

Firms hire labor and capital to produce final goods which are either consumed or invested as new capital goods. They use a simple Cobb-Douglas production technology. The representative firm's problem is:

$$\max_{K,L} K^\alpha (e^{gt+z}L)^{1-\alpha} - rK - wL$$

Where K is the capital hired by the firm, L is the amount of labor it hires, g is the exogenous growth rate of labor-augmenting technology, and z is a stochastic technology shock.

The solution is characterized by the following three equations.

$$r = \alpha Y/K \tag{2.5}$$

$$w = (1-\alpha)Y/L \tag{2.6}$$

$$Y = K^\alpha (e^{gt+z}L)^{1-\alpha} \tag{2.7}$$

Technology is assumed to evolve over time according to the following law of motion.

$$z' = \psi_z z + e_z'; \; e_z' \sim iid(0, \sigma_z^2) \tag{2.8}$$

Government

Each period the government collects revenues and makes payments on two separate accounts. The first is a redistribution of the capital of deceased households over the current population. We assume an equal share for each household regardless of age. Since this is a pure redistribution scheme, the account must balance each period.

$$T' = \frac{\sum_{S=1}^{S} N_S(1-\rho_S)k_S}{\sum_{S=1}^{S} N_S'} \tag{2.9}$$

The second is the social security system, which accumulates a balance over time on a trust fund, denoted H, as illustrated below.

$$H' = H + \sum_{S=E}^{R-1} N_S \tau w \overline{\ell}_S - \sum_{S=R}^{S} N_S b_S \tag{2.10}$$

Benefits are assigned when a household retires at age R and are a function of the average index of monthly earnings (AIME) at retirement. We assume that the benefit is some fraction, θ, of this value.

$$b_R = \theta a_R \tag{2.11}$$

For any individual AIME evolves as a running average over ages E to R according to:

$$a_{S+1}' = \frac{s-E-1}{s-E}a_S + \frac{1}{s-E}w\overline{\ell}_S \text{ for } E \le s \le R-1$$

However, since there is immigration in our model, new individuals who have zero past earnings for purposes of AIME calculations are continually entering the cohort. We take the weighted average of the surviving domestic workers' AIME and zero for immigrant workers when calculating the cohort's new value next year.

$$a'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}} \left[\frac{s-E-1}{s-E}a_s + \frac{1}{s-E}w\bar{\ell}_s \right] \text{ for } E \leq s \leq R-1 \tag{2.12}$$

Once set at retirement benefits remain constant until death, however immigration averaging applies in this case also. New immigrants of retirement age or older receive no benefits.

$$b'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}} b_s \text{ for } s > R \tag{2.13}$$

Market-clearing and Aggregation

The capital and labor market clearing conditions are given by:

$$K = \sum_{s=1}^{S} N_s k_s + H \tag{2.14}$$

$$L = \sum_{s=1}^{S} N_s \bar{\ell}_s \tag{2.15}$$

There is also a goods market clearing condition, $Y + (1 - \delta)K = \sum_{s=1}^{S} c_s + K'$, but it is redundant by Walras Law.

The laws of motion for the demographic parameters are as follows:

$$f_s' = (1 - \psi_{fs})\bar{f}_s + \psi_{fs}f_s + e_{fs}'; e_{fs}'\sim iid(0, \sigma_{fs}^2) \tag{2.16}$$

$$\iota_s' = (1 - \psi_{\iota s})\bar{\iota}_s + \psi_{\iota s}\iota_s + e_{\iota s}'; e_{\iota s}'\sim iid(0, \sigma_{\iota s}^2) \tag{2.17}$$

$$\rho_s' = (1 - \psi_{\rho s})\bar{\rho}_s + \psi_{\rho s}\rho_s + e_{\rho s}'; e_{\rho s}'\sim iid(0, \sigma_{\rho s}^2) \tag{2.18}$$

We can easily consider the special case where these values are constant by setting the variances of the shocks (σ's) the autocorrelations (ψ's) to zero.

Equations (2.1) through (2.18) define the model. There are $S + 1$ exogenous state variables: the cohort populations, $\{N_s\}_{s=1}^{S}$, and the technology shock, z. Since capital prior to age E is assumed to be zero, there are $2(S - E) + 1$ endogenous state variables: the bond holdings for each cohort, $\{k_s\}_{s=E+1}^{S}$, AIME for each cohort from labor force entry until

retirement, $\{a_s\}_{s=E}^{R-1}$, benefits for every cohort thereafter, $\{b_s\}_{s=R}^{S}$, and the balance on the social security trust fund, H.

3. A Stationary Version

Our model as written is non-stationary. Technology has a trend rate of growth, g, and the population may also be growing over time. We can write equations (2.1) & (2.2) in matrix notation.

$$\mathbf{N'} = \mathbf{\Gamma N}\ ;\ \mathbf{\Gamma} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{S-1} & f_S \\ \rho_2 + \iota_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_3 + \iota_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \rho_S + \iota_S & 0 \end{bmatrix}$$

Where $\mathbf{N}$ is the $S \times 1$ vector of cohort populations. We define the total population as $= \mathbf{1}_{1\times S}\mathbf{N}$. The growth rate of the population comes from $N' = (1 + n')N$ and by substitution this is $n' = \frac{\mathbf{1}_{1\times S}\mathbf{\Gamma N}}{\mathbf{1}_{1\times S}\mathbf{N}} - 1$

In order to solve our model using the numerical techniques we propose, it is necessary to transform the non-stationary variables to stationary ones. Some per capita variables, such as consumption and wages, will grow at the long-run rate of g. We transform these variables by defining a stationary version that removes this growth. We denote these transformed variables with a carat (^). $\hat{x} \equiv x/e^{gt}$ for $x \in (\{k_s\}_{s=2}^{S}, \{a_s\}_{s=E}^{R}, \{b_s\}_{s=R}^{S}, \{c_s\}_{s=1}^{S}, w)$

To transform the cohort populations we need to remove a unit root, which we do by dividing by the total population, N. $\hat{x} \equiv x/N$ for $y \in (\{N_s\}_{s=1}^{S}, L)$

Finally some aggregate variable grow at the rate g and also have a unit root. $\hat{x} \equiv x/(Ne^{gt})$ for $y \in (Y, K, H)$

If we assume a within-period utility function, $u(c) = \frac{1}{1-\gamma}(c^{1-\gamma} - 1)$, the transformed equations that define the stationary model are:

$$z' = \psi_z z + e_z'; e_z' \sim iid(0, \sigma_z^2) \tag{3.1}$$

$$f_s' = (1 - \psi_{fs})\bar{f}_s + \psi_{fs}f_s + e_{fs}'; e_{fs}' \sim iid(0, \sigma_{fs}^2) \tag{3.2}$$

$$\iota_s' = (1 - \psi_{\iota s})\bar{\iota}_s + \psi_{\iota s}\iota_s + e_{\iota s}'; e_{\iota s}' \sim iid(0, \sigma_{\iota s}^2) \tag{3.3}$$

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$$\rho_s' = \left(1 - \psi_{\rho s}\right)\bar{\rho}_s + \psi_{\rho s}\rho_s + e_{\rho s}'; \; e_{\rho s}' \sim iid(0, \sigma_{\rho s}^2) \tag{3.4}$$

$$n' = \frac{\mathbf{1}_{1 \times S}(\bar{\Gamma} + \hat{\Gamma})\hat{N}}{\mathbf{1}_{1 \times S}\hat{N}} - 1 \tag{3.5}$$

$$\hat{N}_{s+1}'(1 + n') = \hat{N}_s(\rho_s + \iota_s + \varepsilon_{s+1}') \tag{3.6}$$
$$\text{for } 1 \leq s \leq S - 1$$

$$\hat{N}_1'(1 + n') = \sum_{s=1}^{S}(f_s + v_1')\hat{N}_s \tag{3.7}$$

$$\hat{c}_s = \hat{w}\bar{\ell}_s(1 - \tau) + (1 + r - \delta)\hat{k}_s - (1 + g)\hat{k}_{s+1}' + \hat{b}_s + \hat{T} \tag{3.8}$$
$$\text{for } 1 \leq s \leq S$$

$$\hat{c}_s^{-\gamma} = \beta E\{[\hat{c}_s'(1 + g)]^{-\gamma}(1 + r' - \delta)\} \tag{3.9}$$
$$\text{for } 1 \leq s \leq S - 1$$

$$r = \alpha \hat{Y}/\hat{K} \tag{3.10}$$

$$\hat{w} = (1 - \alpha)\hat{Y}/\hat{L} \tag{3.11}$$

$$\hat{Y} = \hat{K}^\alpha (e^z \hat{L})^{1-\alpha} \tag{3.12}$$

$$\hat{T}' = \frac{\sum_{s=1}^{S} \hat{N}_s(1 - \rho_s)\hat{k}_s}{(1 + n')\sum_{s=1}^{S} \hat{N}_s'} \tag{3.12}$$

$$\hat{H}'(1 + g) = \hat{H} + \sum_{s=E}^{R-1} \hat{N}_s \tau \hat{w}\bar{\ell}_s - \sum_{s=R}^{S} \hat{N}_s \hat{b}_s \tag{3.14}$$

$$\hat{a}_{s+1}'(1 + g) = \frac{s - E - 1}{s - E}\hat{a}_s + \frac{1}{s - E}\hat{w}\bar{\ell}_s \tag{3.15}$$
$$\text{for } E \leq s \leq R - 1$$

$$\hat{b}_R = \theta \hat{a}_R \tag{3.16}$$

$$\hat{b}_{s+1}'(1 + g) = \hat{b}_s \tag{3.17}$$
$$\text{for } s > R$$

$$\hat{K} = \sum_{s=1}^{S} \hat{N}_s \hat{k}_s + \hat{H} \tag{3.18}$$

$$\hat{L} = \sum_{s=1}^{S} \hat{N}_s \bar{\ell}_s \tag{3.19}$$

4. Calibration and Steady States

We have the following set of parameters we must calibrate in order to simulate the model.

S	maximum age in periods
E	period workers enter the labor force
R	period workers retire

$\{\bar{f}_s\}_{s=1}^S$	average fertility rates by age
$\{\bar{\iota}_s\}_{s=2}^S$	average immigration rates by age
$\{\bar{\rho}_s\}_{s=2}^S$	average survival rates by age
$\{\bar{\ell}_s\}_{s=1}^S$	effective labor endowment by age
τ	payroll tax rate
δ	capital depreciation rate
β	subjective discount factor
g	growth rate of technology
γ	coefficient of relative risk aversion
α	capital share in GDP
θ	pension benefits as percent of AIME

In addition we have parameters governing the stochastic processes.

$\psi_z, \{\psi_{fs}, \psi_{\iota s}, \psi_{\rho s}\}_{s=1}^S$	autocorrelations
$\sigma_z^2, \{\sigma_{fs}^2, \sigma_{\iota s}^2, \sigma_{\rho s}^2\}_{s=1}^S$	variances

We set the number of periods in our model and interpret the period so that S periods corresponds to 100 years. We assume agents become financially independent at and enter the labor force at age 16, which gives $E = \text{round}(\frac{16}{100}S)$. We assume retirement occurs at age 65 so that $R = \text{round}(\frac{65}{100}S)$. The depreciation rate is set to correspond to an annual rate of 10%, $\delta = 1 - (1 - 0.1)^{100/S}$. Similarly, β is chosen to yield an annual rate of time preference of approximately 2%, $\beta = .98^{100/S}$. And g is chosen to yield an annual growth rate of technology of 1.5%, $g = (1 + 0.015)^{100/S}$.

The capital share in GDP (α) is set to 0.33. γ is the intertemporal elasticity of substitution and we set this to 1.0, which yields logarithmic utility. The benefit to AIME ratio (θ) is set to .186, the average ratio of OASDI benefits per retiree to the average worker's wage for the years 2001 through 2009. The payroll tax rate (τ) is chosen to make total social security benefits and taxes equal in the steady state.

Effective labor supply, fertility rates, survival rates and immigration rates by age are estimated using data from a variety of sources. Data on effective labor comes from the Bureau of Labor Statistics' Current Population Survey. Data on immigration rates come

from the US Census Bureau. Fertility rates come from Nishiyama & Smetters (2007). Cumulative survival rates come from the Center for Disease Control's (CDC) mortality tables. We fit polynomials to the data by age. For fertility and immigration we fit the number of births or immigrants of a certain age as a percent of the population of that age.

Data for effective labor supply comes from quarterly earnings data for 2001 through 2010. We use earnings because our effective labor includes both hours worked and the productivity of the worker. Since wage rates should be proportional to productivity, we can simply use earnings which is hours worked times the wage rate per hour. We normalize so that the average earnings over the ages reported is one. We then fit earnings by age to the average age of the cohort using a 6th-order polynomial in the age. Since these polynomials are ill-behaved at the ends, we interpolate exponentially to get better fit there. Figure 1 shows the data and the fitted curve. When we simulate we choose the size of a period in years, and use this fitted curve to get effective labor for each cohort.

Data for immigration is available from 2005 detailing the number of those who immigrated between 2000 and 2005. Immigrants are grouped into cohorts of five years. We calculate the number of immigrants as a percentage of the US population in 2000. We then fit this percentage by age to the average age of the cohort using a 6th-order polynomial in the age. We interpolate linearly at the ends. Figure 2 shows the data and the fitted curve. When we simulate we choose the size of a period in years, and use this fitted curve to get immigration rates for each cohort.

For fertility rates the data are available in 5 year cohorts as well. Fertility rates below age 15 and above age 50 are effectively zero. We proceed as above and fit this data with a 3rd-order polynomial in age. Again we interpolate, but only on the upper end. The data and fitted curve are shown in Figure 3.

For survival rates, we fit data on the cumulative probability of surviving to a particular age. We infer the conditional survival rates from this fitted polynomial. Data are available for 10-year cohorts. We fit this with a 3rd-order polynomial and interpolate on the upper end so that mortality reaches 100% at age 100. The data and fitted curve are shown in Figure 4.

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With the model calibrated we can easily solve for the steady state. We do so using numerical techniques. The steady state is summarized in Table 1 and in Figure 5.

5. Solution and Simulation

We propose solving and simulating our model in the same way that many dynamic stochastic general equilibrium (DSGE) models with infinitely-lived agents are solved and simulated by linear approximation. To see the parallels we first outline the methodology for the infinitely-lived representative agent case.

Consider a simple infinitely-lived agent's problem.

$$V(k; z) = \max_{k'} u(c) + \beta E\{V(k'; z)\}$$

With $c = w\bar{\ell} + (1 + r - \delta)k - k', y = y(k, z), r = y_k(k, z) \,\&\, w = y(k, z) - y_k(k, z)k.$

The Euler equation in this case is:

$$u^c(c) = \beta E\{u^c(c')(1 + r' - \delta)\} \tag{5.1}$$

The single endogenous state variable is k and we have assumed there is a single technology shock, z. To solve this model we first log-linearize our Euler equation about the model's steady state. We can write this in the form below, where the tildes ($\sim$) denote log-deviations from steady state values.

$$E_t\{T + F\tilde{k}_{t+1} + G\tilde{k}_t + H\tilde{k}_{t-1} + L\tilde{z}_{t+1} + M\tilde{z}_t\} = 0 \tag{5.2}$$

Where F, G, H, L & M are coefficients that are functions of parameters and steady state values. When linearizing about the steady state, T will be zero.

Assuming a log linear law of motion for z, $\tilde{z}_{t+1} = (1 - N)\bar{z} + N\tilde{z}_t + e_{t+1}$, and assuming that the transition function, $k_{t+1} = \phi(k_t, z_{t+1})$, can also be written in log-linear form we can find its coefficient values.

$$\tilde{k}_{t+1} = P\tilde{k}_t + Q\tilde{z}_{t+1} + U \tag{5.3}$$

The techniques for finding the numerical values of P & Q are well-known and involve solving a quadratic in P.[1] Solution techniques for U are less commonly used, but easy to

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[1] See Uhlig (1999) or Christiano (2002).

derive. They can be shown to yield a U equal to zero when linearizing about the steady state.

Next, consider an OLG model with a similar setup. An age s agent solves the following problem.

$V_s(k_s, \mathrm{z}) = \max_{k_{s+1}}, u(c_s) + \beta E\{V_{s+1}(k_{s+1}', \mathrm{z}')\}$

With $c_s = w\bar{\ell}_s + (1 + r - \delta)k_s - k_{s+1}'$, $K = \sum_{i=1}^{J} N_s k_s$, $y = f(k; \mathrm{z})$, $r = f_k(k; \mathrm{z})$, $w = f(k; \mathrm{z}) - f_k(k; \mathrm{z})k$

The Euler equation in this case is:

$u^c(c_s) = \beta E\{u^c(c_{s+1}')(1 + r' - \delta)\}$

If we set up and solve each agent's problem and then stack the variables for each agent such that $\mathbf{x} \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_S \end{bmatrix}$, we get the following matrix representation of the model[2], where bold variables indicate matrices.

$\mathbf{V}(\mathbf{k}, \mathrm{z}) = \max_{\mathbf{k}'}, \mathbf{u}(\mathbf{c}) + \beta \Delta E\{\mathbf{V}(\mathbf{k}', \mathrm{z}')\}$

with $\quad \mathbf{c} = w\bar{\ell}(1 - \tau) + (1 + r - \delta)\mathbf{k} - \Delta\mathbf{k}'$, $\quad K = \mathbf{1}_{1\times S} \cdot (\mathbf{N} \circ \mathbf{k})$, $\quad y = y(K; \mathrm{z})$,

$r = y_k(K; \mathrm{z})$, $w = y(K; \mathrm{z}) - y_k(K; \mathrm{z})K$, $\Delta \equiv \begin{bmatrix} \mathbf{0}_{(S-1)\times 1} & \mathbf{I}_{(S-1)\times(S-1)} \\ 0 & \mathbf{0}_{1\times(S-1)} \end{bmatrix}$. $\mathbf{V}(\mathbf{k}, \mathrm{z})$ and $\mathbf{u}(\mathbf{c})$ are $S\times 1$ vector-valued functions.

The stacked Euler equations are:

$\mathbf{u}^c(\mathbf{c}) = \beta\Delta E\{\mathbf{u}^c(\mathbf{c}')(1 + r' - \delta)\}$ \hfill (5.3)

where $\mathbf{u}^c$ is an $S\times 1$ vector of the derivatives of $\mathbf{u}(\mathbf{c})$ with respect to the sth element of $\mathbf{c}$. Note that the final Sth row is dropped since the S aged agent has no Euler equation.

We can solve and simulate this model just as we do the DSGE model above.

We write the log-linearized versions of the Euler equations in the following form:

$E_t\{\mathbf{T} + \mathbf{F}\tilde{\mathbf{k}}_{t+1} + \mathbf{G}\tilde{\mathbf{k}}_t + \mathbf{H}\tilde{\mathbf{k}}_{t-1} + \mathbf{L}\tilde{z}_{t+1} + \mathbf{M}\tilde{z}_t\} = 0$ \hfill (5.4)

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[2] Note that a $'$ always denotes next period, not a transpose. A transpose is denoted with a $\mathbf{T}$ superscript, instead.

We use the same numerical techniques as above to solve for the matrices $\mathbf{P}$ & $\mathbf{Q}$ in the log-linearized transition functions.

$$\tilde{\mathbf{k}}_{t+1} = \mathbf{P}\tilde{\mathbf{k}}_t + \mathbf{Q}\tilde{z}_{t+1} + \mathbf{U} \tag{5.5}$$

To simulate our particular model we use the linearized transition functions for our stationary model laid out in section 3.[3]

$$\tilde{\mathbf{X}}_{t+1} = \mathbf{P}\tilde{\mathbf{X}}_t + \mathbf{Q}\tilde{\mathbf{Z}}_{t+1} + \mathbf{U} \tag{5.6}$$

Where $\mathbf{X}_t = [\{k_{s,t+1}\}_{s=2}^S \ \{a_{s,t+1}\}_{s=E+1}^R \ \{b_{s,t+1}\}_{s=R}^S]$ and $\mathbf{Z}_t = [z_t \ \{f_{s,t} \ \iota_{s,t} \ \rho_{s,t}\}_{s=1}^S]$.

Along with the exogenous laws of motion defined by equations (3.1) – (3.4) which we rewrite collectively as:

$$\tilde{\mathbf{Z}}_{t+1} = \mathbf{N}\tilde{\mathbf{Z}}_t + \mathbf{e}_{t+1} \tag{5.7}$$

We begin our simulation with initial conditions for the log-deviations of the state variables, $\mathbf{X}_t$ & $\mathbf{Z}_t$, from their steady state values. We also draw a series of random shocks for the values of $\mathbf{e}_t$ in each period. Equations (5.5) & (5.6) allow us to generate a time series for the log-deviations of our state variables from their steady states.

We can reconstruct the stationary versions of state variables by treating them as percent deviations using $\hat{x}_t = \bar{x}e^{\tilde{x}_t}$. We can also construct the total population using an initial value, the formula $N_{t+1} = (1 + n_{t+1})N_t$, and by noting that n_{t+1} is a function of our stationary state variable by equation (3.5).

Finally, we can construct non-stationary variables by putting in the appropriate trend and/or unit root, $x_t = e^{gt}\hat{x}_t$, $x_t = N_t\hat{x}_t$ or $x_t = N_t e^{gt}\hat{x}_t$. Once we have a time-series for all state variables in the non-stationary model, we can find the value of any other variable of interest by using the appropriate structural equation(s) from section 2.

The methodology above works well for simulations where the state of the economy deviates only in a neighborhood about the steady state. However, our model is dynamically unstable. This means that even if we start out at the model's steady state values, the

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[3] We use equations (3.9), (3.15), (3.16) & (3.17) as the dynamic equations which are linearized. Equations (3.1) – (3.4) define the exogenous laws of motion. The remaining equations are used as definitions.

stochastic shocks will drive us away from that point and the model will explode thereafter. We need a simulation technique that will be accurate when we are far from the steady state.

One technique that fits the bill is to linearize about the current state of the economy rather than around the steady state. We can use equations (5.4), (5.5) & (5.7), but we reinterpret the tilde as the deviation of the variable from its value now, rather than its value in the steady state. We rewrite these equations noting that the coefficients will now be time dependent since we linearize about a different point each period.

$$E_t\{\mathbf{T}_t + \mathbf{F}_t\widetilde{\mathbf{X}}_{t+1} + \mathbf{G}_t\widetilde{\mathbf{X}}_t + \mathbf{H}_t\widetilde{\mathbf{X}}_{t-1} + \mathbf{L}_t\widetilde{\mathbf{Z}}_{t+1} + \mathbf{M}_t\widetilde{\mathbf{Z}}_t\} = 0 \tag{5.8}$$

$$\widetilde{\mathbf{Z}}_{t+1} = \mathbf{N}\widetilde{\mathbf{Z}}_t + (\mathbf{N} - \mathbf{I})(\mathbf{Z}_t - \bar{\mathbf{Z}}) + \mathbf{e}_{t+1} \tag{5.9}$$

$$\widetilde{\mathbf{X}}_{t+1} = \mathbf{P}_t\widetilde{\mathbf{X}}_t + \mathbf{Q}_t\widetilde{\mathbf{Z}}_{t+1} + \mathbf{U}_t \tag{5.10}$$

In this case the matrices $\mathbf{T}_t$ and $\mathbf{U}_t$ will generally not be zero. Since the current state is $(\mathbf{X}_t, \mathbf{Z}_t)$ when we move to next period this becomes $(\mathbf{X}_{t-1}, \mathbf{Z}_{t-1})$. $\mathbf{Z}_t$ is found immediately by using (5.5). So we linearize about the point $(\mathbf{X}_{t-1}, \mathbf{Z}_t)$. This means (5.9) can be rewritten as:

$$\widetilde{\mathbf{X}}_{t+1} = \mathbf{U}_t \tag{5.11}$$

$\mathbf{U}_t$ can be shown to be:

$$\mathbf{U}_t = -(\mathbf{F}_t + \mathbf{G}_t)^{-1}[\mathbf{T}_t + \mathbf{L}_t(\mathbf{N} - \mathbf{I})(\mathbf{Z}_t - \bar{\mathbf{Z}})] \tag{5.12}$$

Simulation proceeds by first setting the values of the initial state. As one simulates each period sequentially:

- (5.9) gives the next value for $\mathbf{Z}_t$.
- One solves for the values of $\mathbf{P}_t, \mathbf{Q}_t, \& \mathbf{U}_t$ by linearizing about $(\mathbf{X}_{t-1}, \mathbf{Z}_t)$.
- (5.10) gives the next value for $\widetilde{\mathbf{X}}_t$
- (5.11) gives $\mathbf{X}_t = \mathbf{X}_t + \mathbf{U}_t$.
- One then proceeds to the next period.

6. Policy Experiments

With the basic methodology in place, we are now ready to proceed with simulation of the model. We first simulate a baseline model where we calibrate the initial state of the

economy to match the current situation in the US. We focus on the time-path of the Social Security trust fund, H_t. We then consider a policy change, resimulate the model, and compare the resulting time-path with the baseline.

Table 1 reports the baseline model's calibrated parameters and the steady state value of key aggregate variables. Figure 5 plots the values of age-specific variables against age.

As we have noted, however, the model is unstable and will rarely generate these values. In order to simulate the model we need to first choose a starting state. Out state is defined by: *S* values for each population cohort, 1 value for the aggregate productivity, *S-E* values for asset holdings of each cohort, *R-E*-1 values for the average index of monthly earnings (AIME) for each cohort of workers, *S-R*+1 values for the benefits paid to each cohort of retirees, and 1 value for the trust fund. We fit the initial distribution of the population to match that of the US population for 2010. We also assume that the technology shock is one standard deviation below it mean, due to the effects of a severe recent recession. We set AIME values at twice their steady state values. In effect, we are assuming that the social security system has promised a level of benefits to current workers that is twice what it will offer in the steady state. Since the steady state is one where total tax revenues match total benefits exactly, this is not unreasonable as it may seem. We assume that social security benefits of current retirees are at their steady state values. Lastly we assume that the initial value of the trust fund is 16.3% of steady state GDP.

We run 1000 Monte Carlo simulations of the economy from this starting point. Figure 7 plots the time path of a zero shock simulation along with 90% confidence bands from the Monte Carlos. The simulations show that the trust fund rises gradually until it peaks in 2024 and then falls explosively in a negative direction. The trust fund turns negative for the first time in 2048. The social security surplus (taxes minus benefits) starts off slightly positive, quickly becomes negative, and reaches a minimum in year 2042. At this point it begins to rise again and is significantly negative by the end of the simulation in year 2086.

We compare this baseline time path with the following policies:

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- An increase in immigration rates across the board.

- Changing immigration rates weighted toward older immigrants.

- A reduction in benefits by 10%.

- An increase in the retirement age to 70.

For the case of increased immigration we double the immigration percentages. The resulting steady state values are reported in Table 2. Figure 8 plots the time path of this simulation and the 90% confidence bands for this case. The trust fund is substantially lower under this scenario. However, the net surplus is substantially higher. Intuitively, the increase in immigration raises tax revenues, but the increased labor force also raises the marginal product of capital and, thus, the interest rate. Since these effects do not become large until after the trust fund has a negative balance, the increased interest costs dominate the increased tax revenues, causing the trust fund to fall more rapidly than in the baseline case.

We also consider a case where immigration rate are shifted so that older workers immigrate more and younger workers immigrate less. Since older immigrants pay taxes for fewer years, they are eligible for fewer benefits that younger immigrants. This could move the system in the direction of long-run solvency. Table 3 and figure 9 illustrate this case. Here the trust fund is substantially less negative in the long-run, but still explodes downward. However, we note that the 90% confidence bands indicate that at least 5% of the time the trust fund exploded in a positive direction. The net surplus is much closer to zero on average and remains roughly balanced even after 75 years.

By way of comparison we also consider cases where benefits are reduced. We consider lowering benefits by 10% immediately and plot the time paths in figure 10. Finally, figure 11 shows the effect of immediately raising the retirement age to 70. In both these cases we find the trust fund never drops into the negative range and the system is unstable in a positive direction.

7. Conclusions

This paper has presented an OLG model with relatively short periods. Rather than solve the model exactly, we have linearized it. This allows us to solve and simulate models with much greater dimensionality that we could by solving exactly using either analytical or numerical methods.

Our model still suffers from the curse of dimensionality, however. For example as the size of the periods in the model get smaller, the number of cohorts rises. The number of state variables in the model is $3S - 2E + 1$. So as the number of cohorts rises, so does the state space. With large enough state spaces the computation of the linear coefficients **P** & **Q** in the transition function becomes computationally burdensome.

A model with idiosyncratic shocks to members of cohorts would be intractable with our solution method. For example, a ten-period-lived-agent model with only 2 values for a binary idiosyncratic shock would give $2^{10} - 1 = 1023$ different agents of various ages, whereas our current model with hundred-period-lived agents has 100 different agents.

Despite its reliance on a representative agent for each cohort our model does yield some useful results concerning immigration. First, expanding immigration across the board does not lead to better long-run outcome for the trust fund. Targeting older immigrants may be a better option. However, the fundamental instability of the system makes long-run predictions very imprecise.

Any Social Security system that defines fixed benefits while relying on stochastic tax revenues will be subject to this instability. A more appropriate arrangement would be for benefits to be somehow dependent on the state of the economy. This seems like a fruitful area for future research.

Figure 1

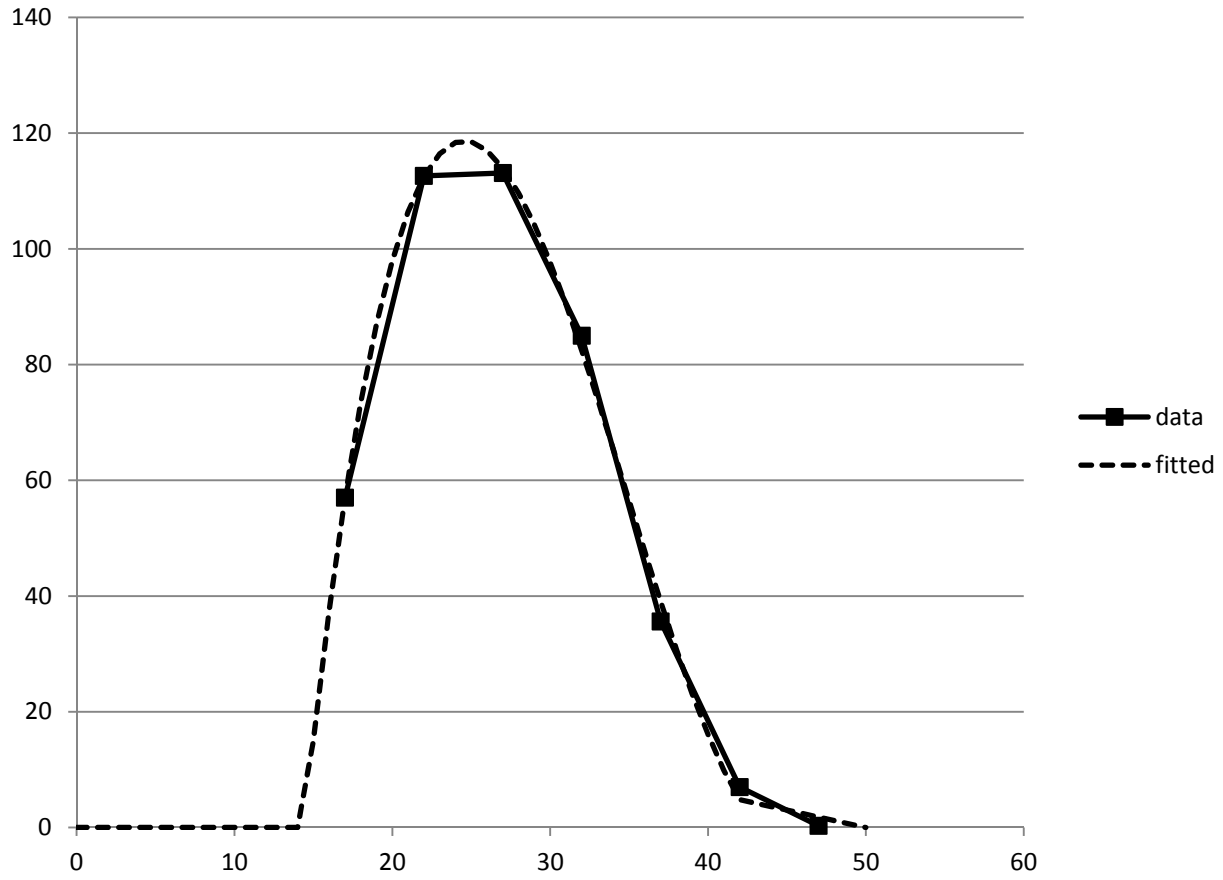**Data and Fitted Curve for Effective Labor by Age[4]**
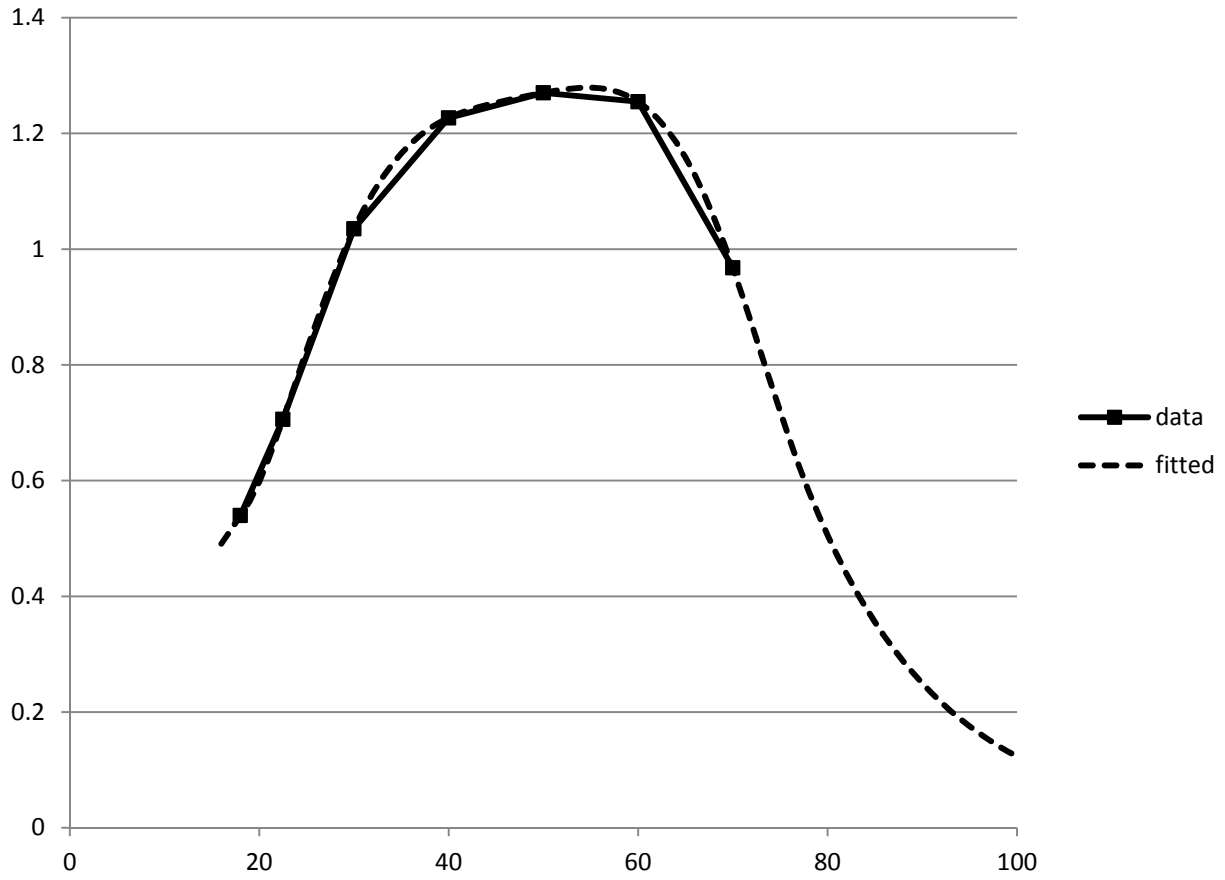
Figure 2

Data and Fitted Curve for Immigration Rates by Age[5]

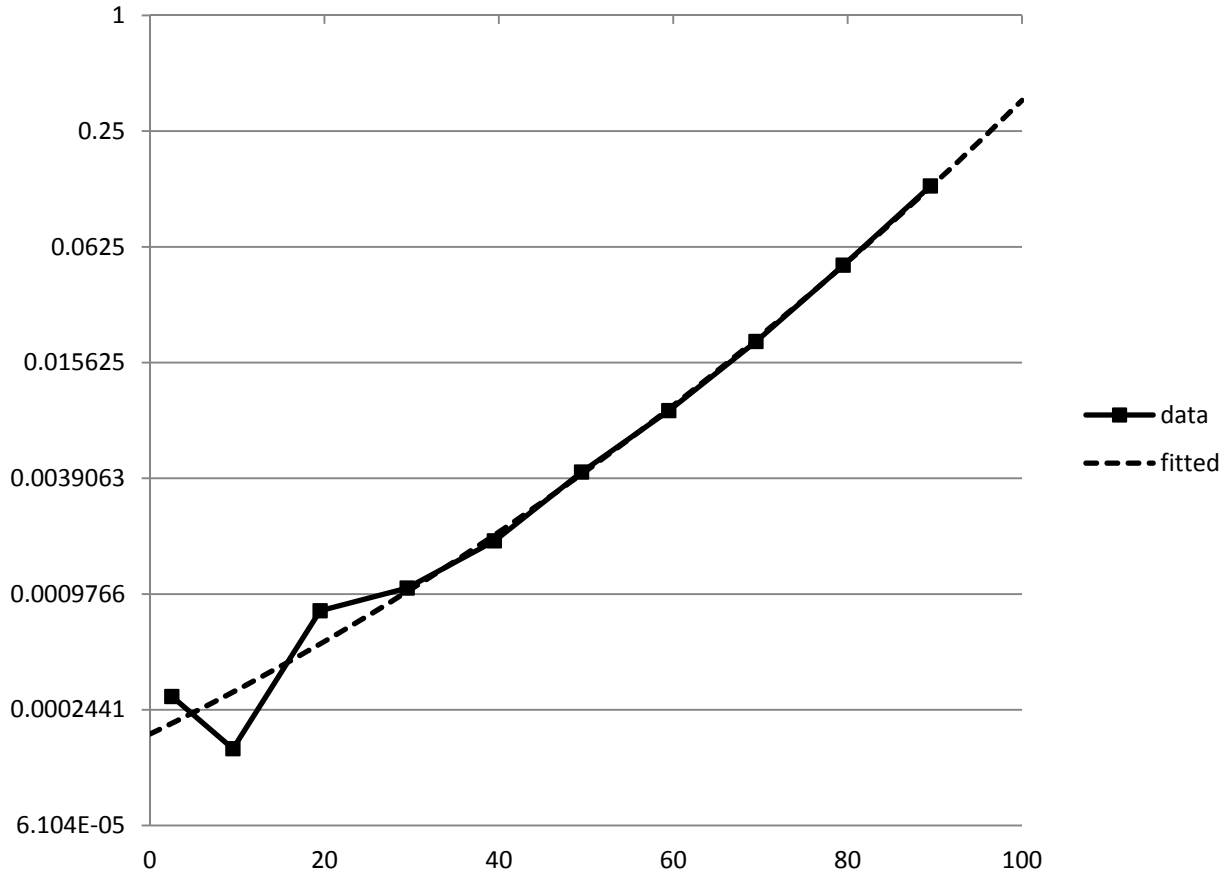**(immigration rates are over a 5-year period)**
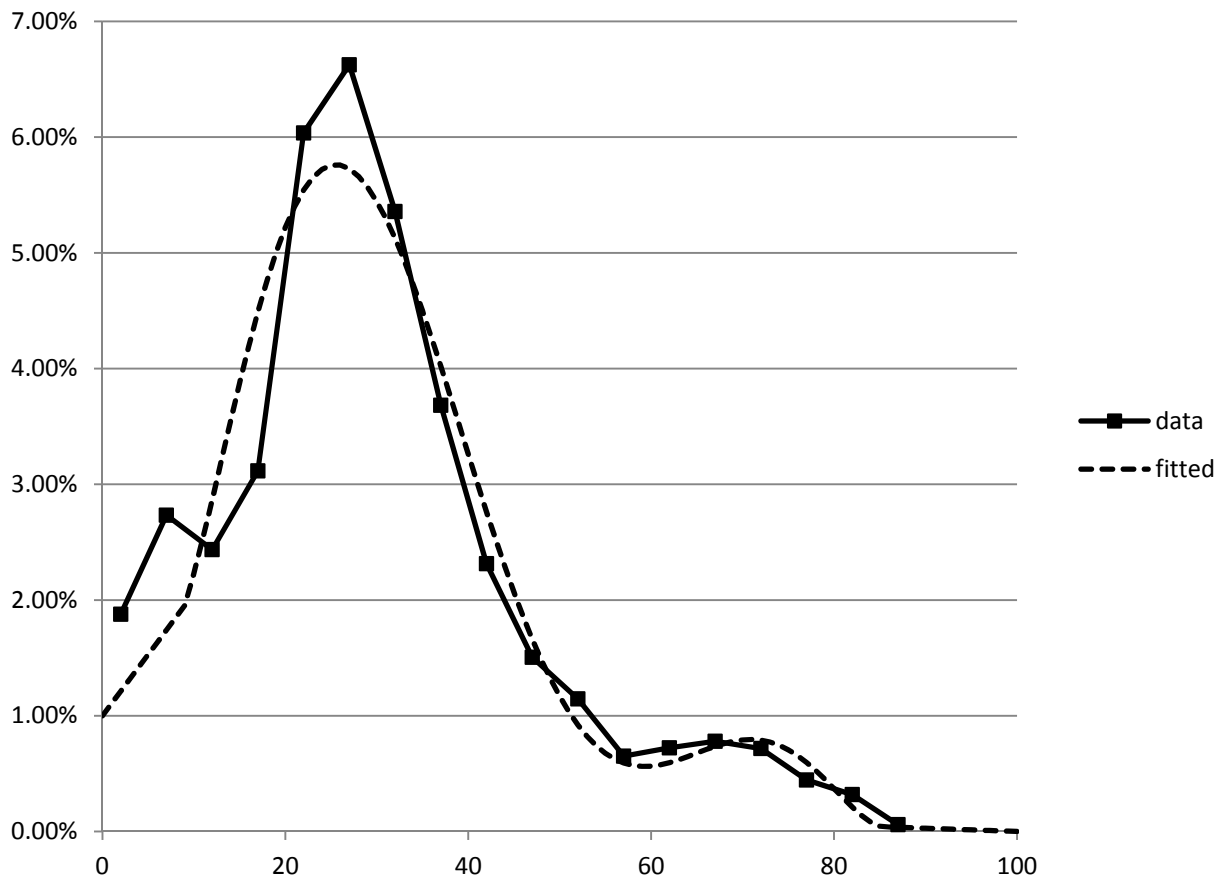
[5] Data are from the US Census Bureau.

Figure 3

Data and Fitted Curve for Fertility Rates by Age[6]

(births per 1000 for females of indicated age per year)

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[6] Data are from Nishiyama (2004).

Figure 4

Data and Fitted Curve for Conditional Hazard Rates by Age[7]

(vertical scale is logarithmic)

[7] Data are from the US Center for Disease Control's mortality tables.

Figure 5

Steady State Values of Selected Variables by Age
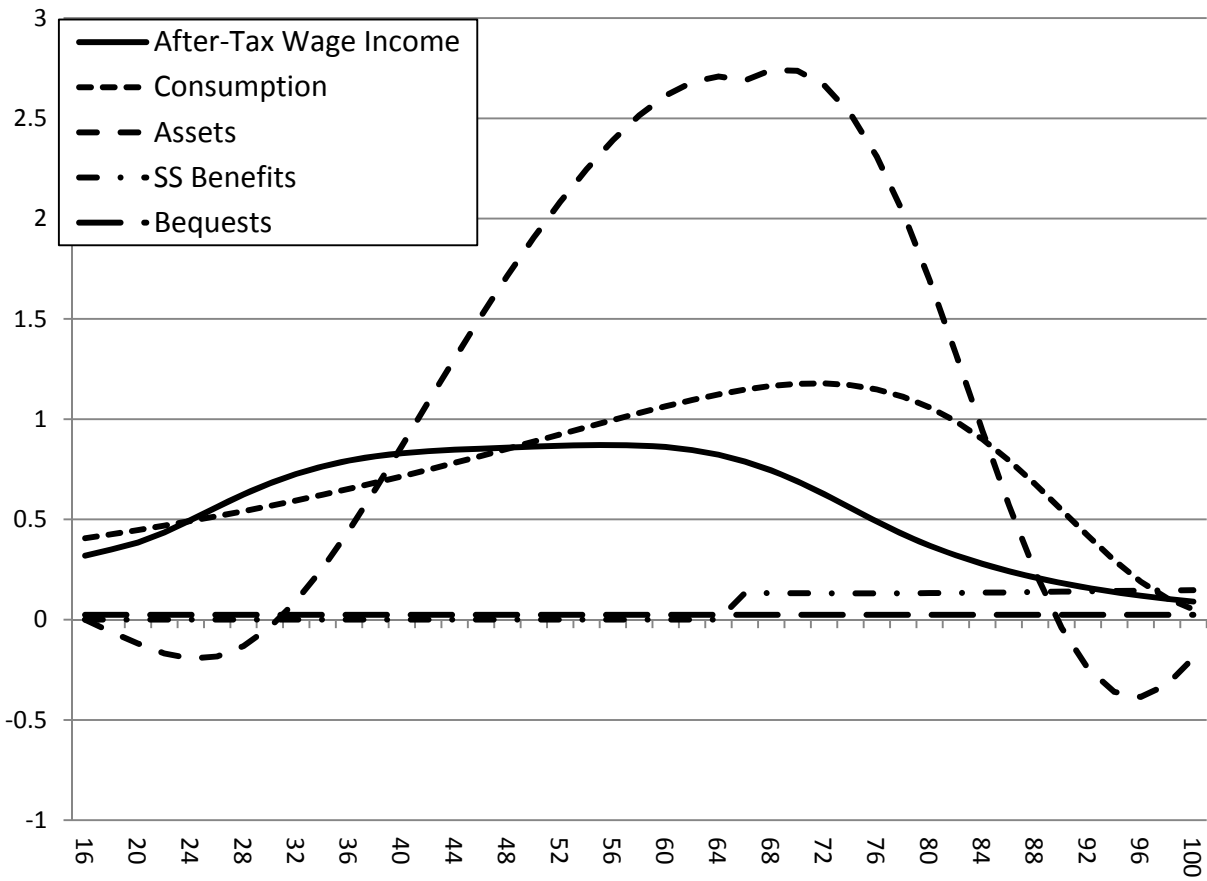
Figure 6

Steady State and Starting Distributions of the Population by Age
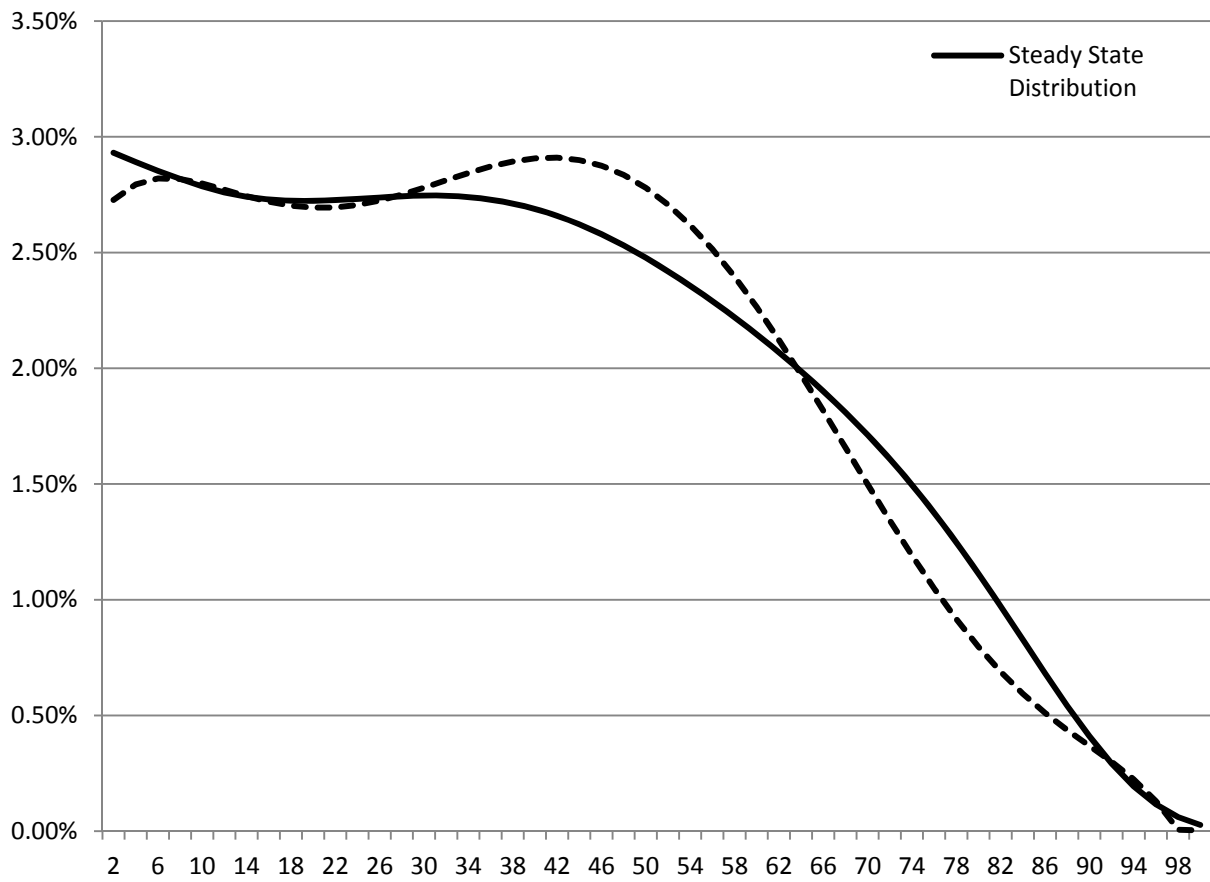
Figure 7

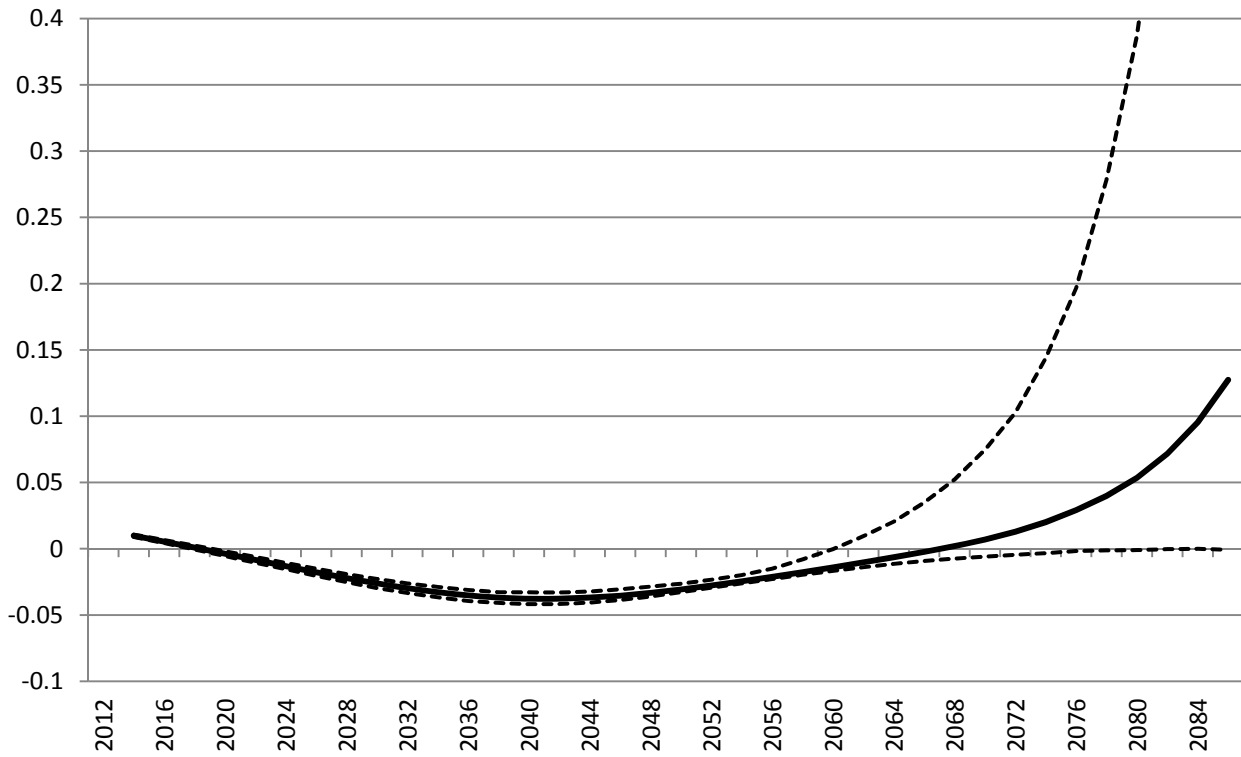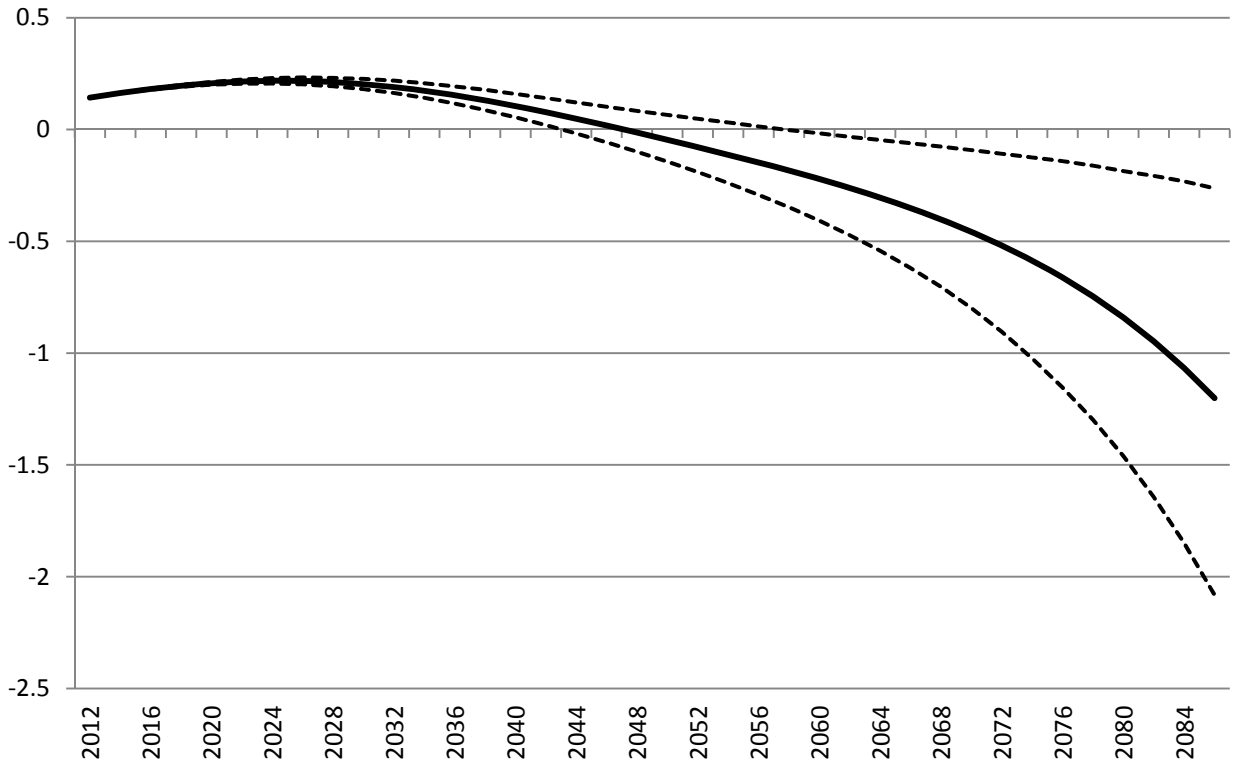**Time Paths for the Trust Fund & Social Security Surplus in the Baseline Case**

Figure 8
Time Paths for the Trust Fund & Social Security Surplus in the Doubled Immigration Rates Case
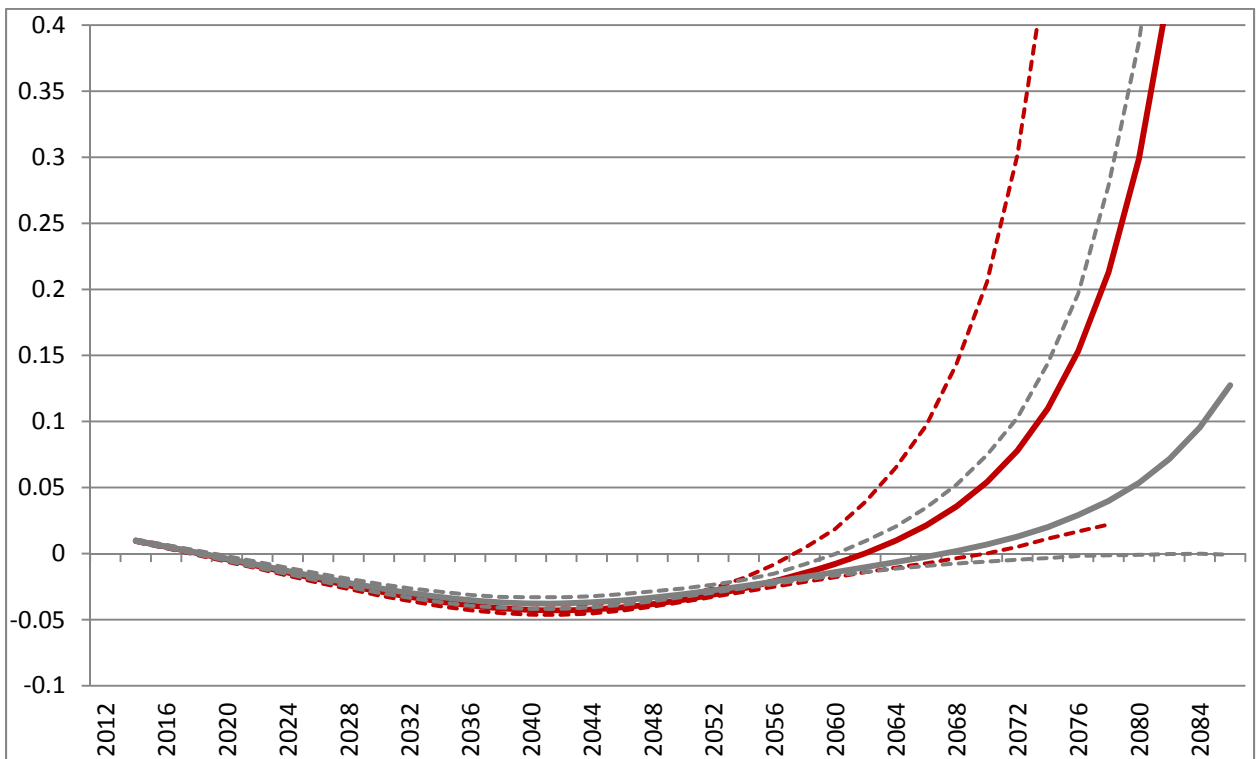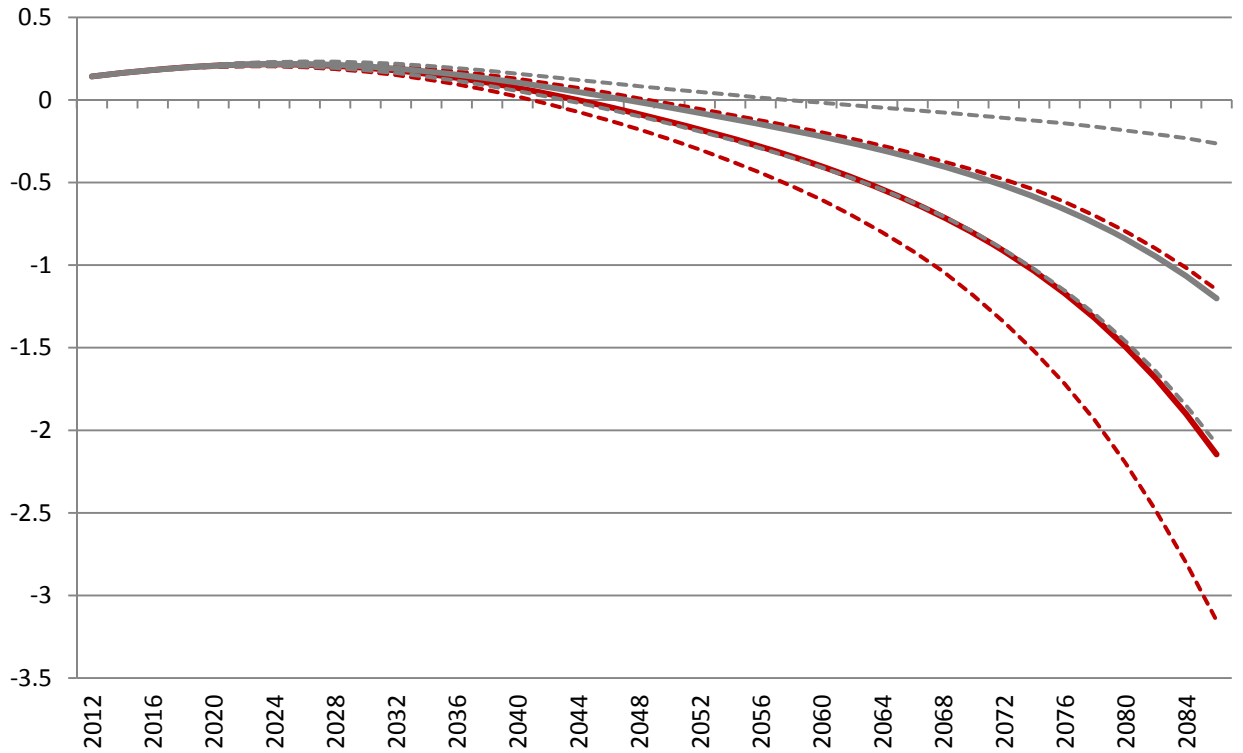
Figure 9
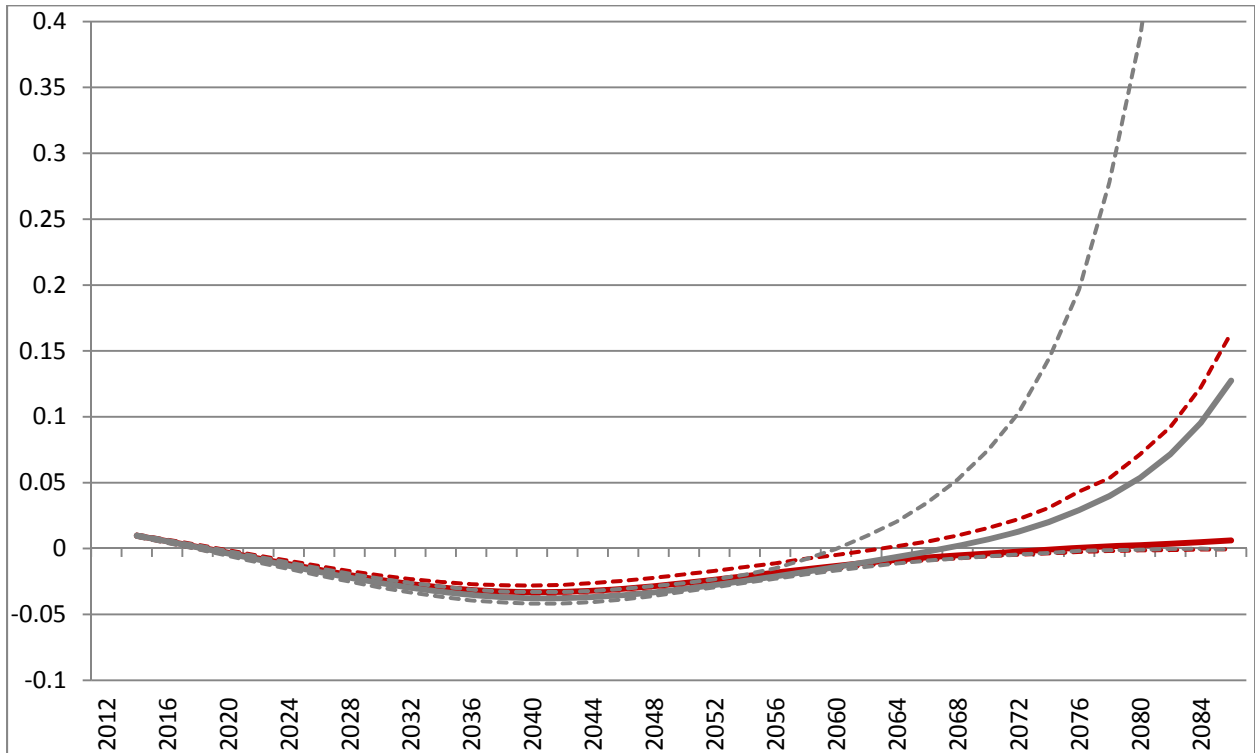**Time Paths for the Trust Fund & Social Security Surplus in the Skewed Immigration Rates Case**
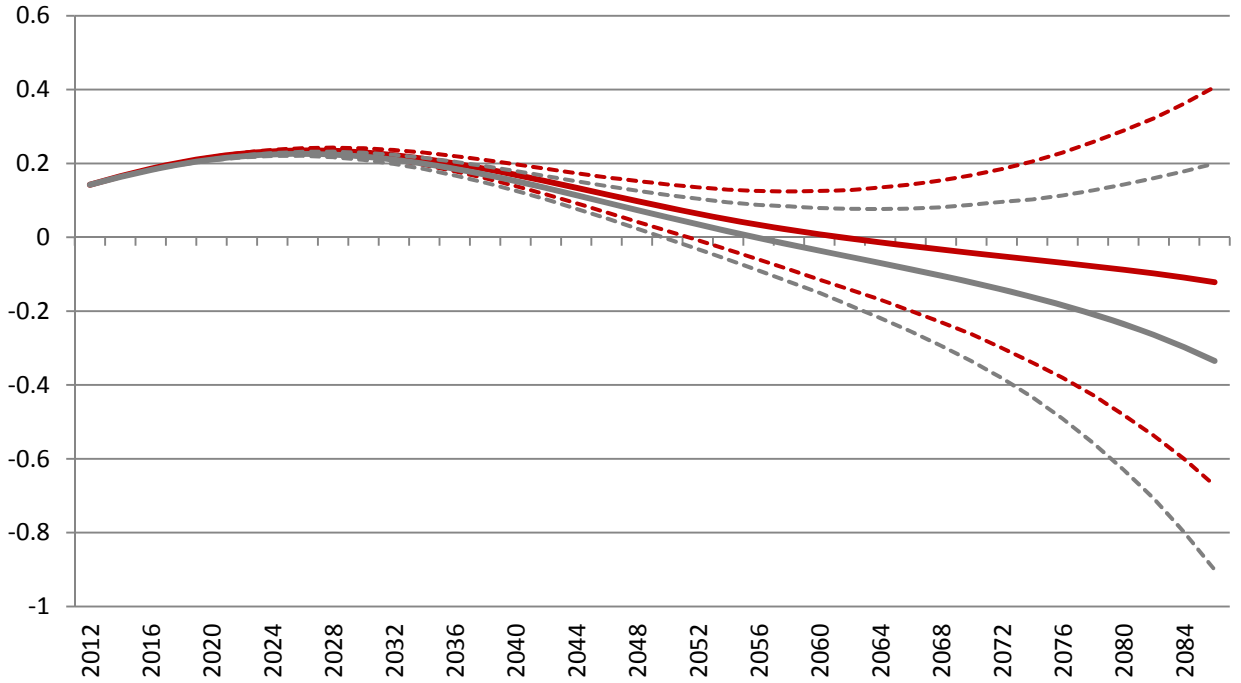
Figure 10
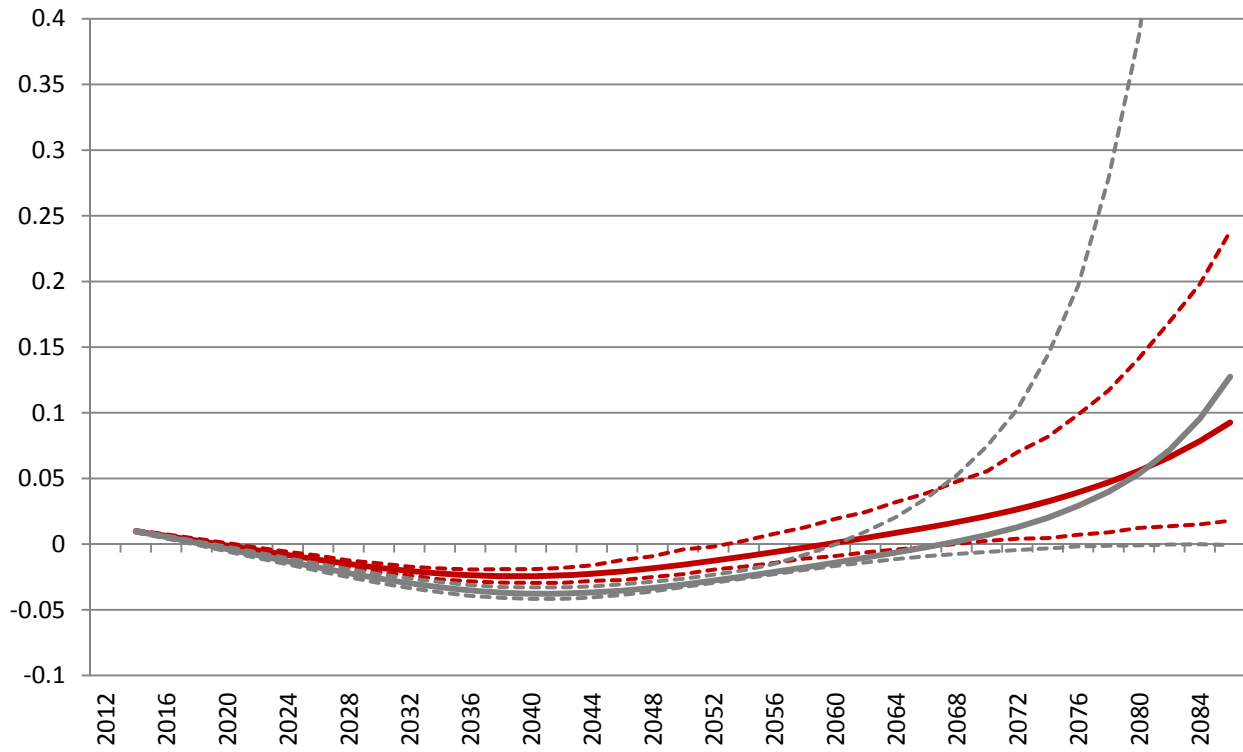**Time Paths for the Trust Fund & Social Security Surplus in the Reduced Benefits Case**
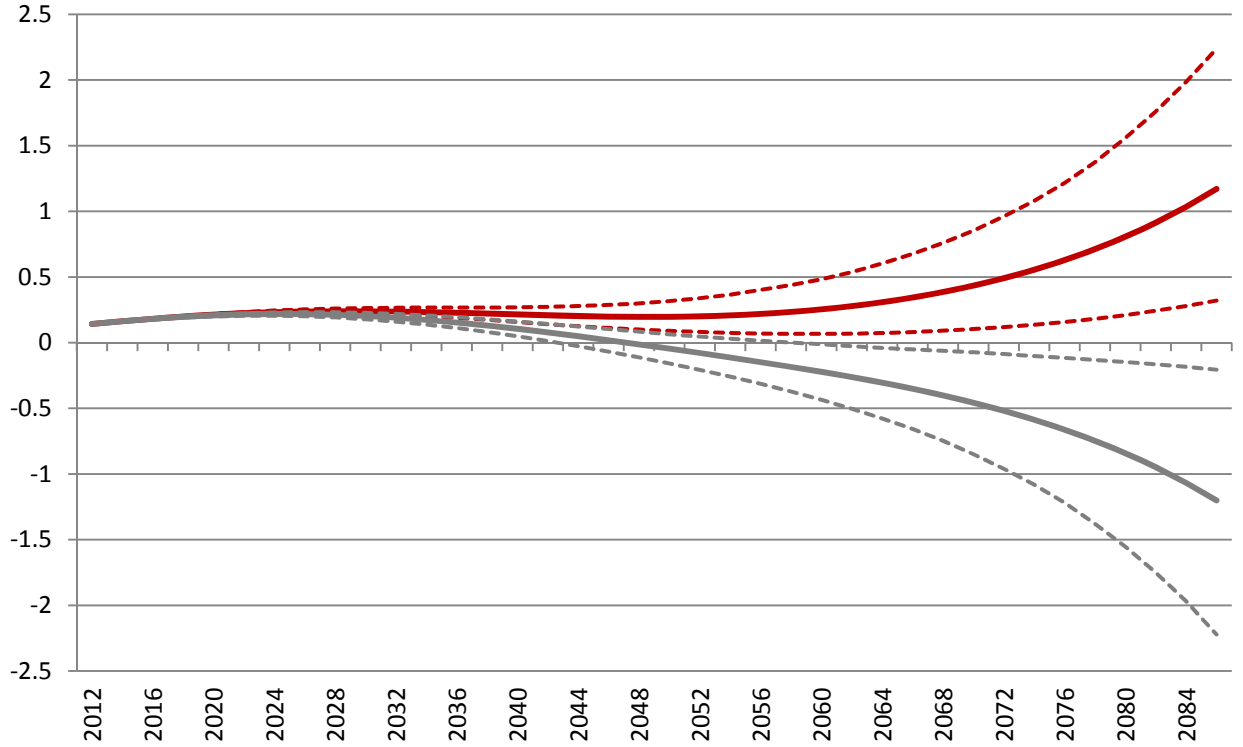
Figure 11
Time Paths for the Trust Fund & Social Security Surplus in the Increased Retirement Age Case
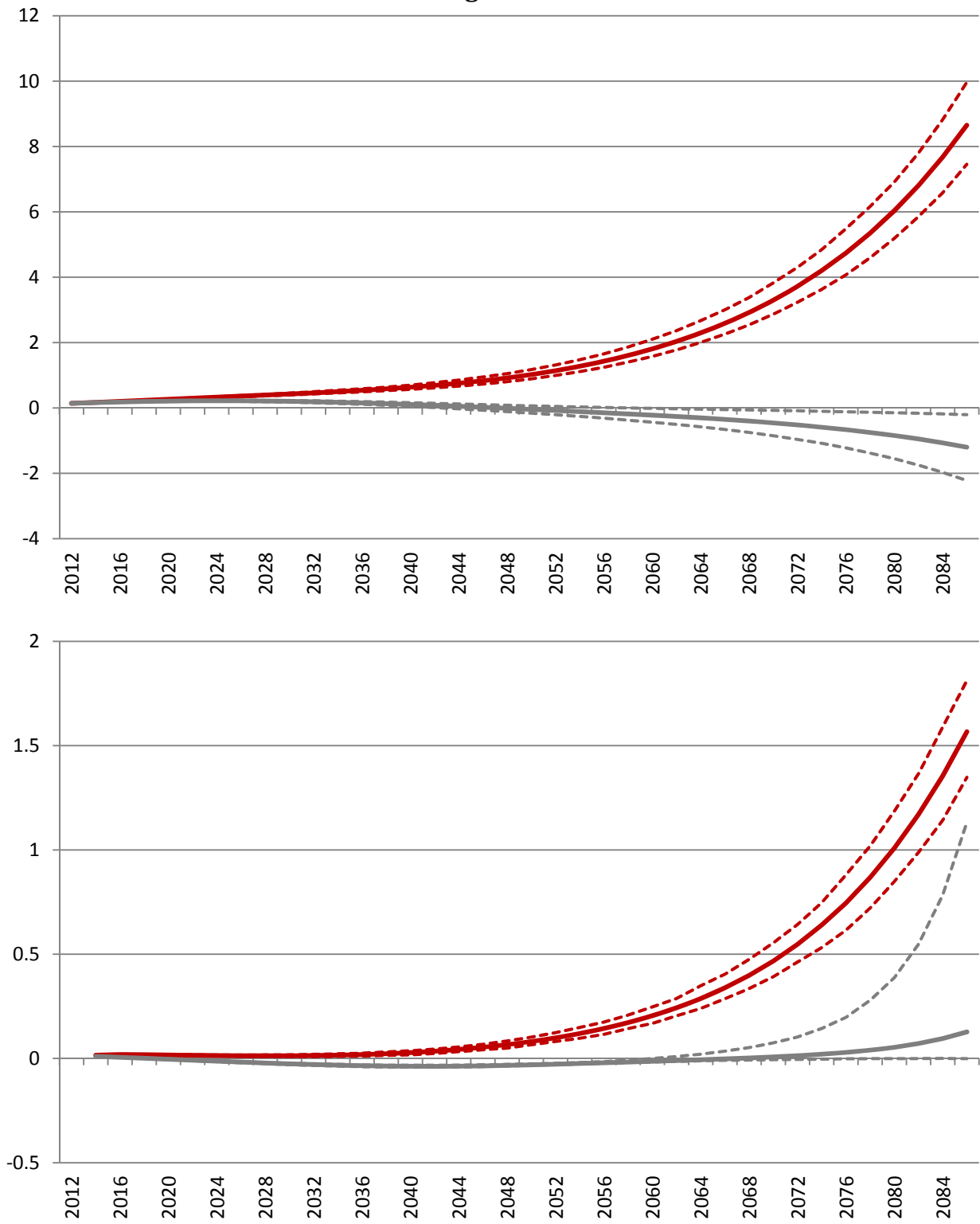
Table 1
Baseline Calibration & Selected Steady State Values

α	0.33	$\bar{K}$	0.9379
γ	1	$\bar{H}$	0.0000
$\bar{g}^{*}$	0.005	$\bar{Y}$	0.8380
δ^{*}	0.1	$\bar{C}$	0.2784
β^{*}	0.98	$\bar{I}$	0.2180
θ	0.2	$\bar{L}$	0.7927
S	50	$\bar{r}^{*}$	0.1379
		$\bar{w}$	0.7082
		$\bar{T}$	0.0242
		$\bar{B}$	0.0218
		$\bar{n}^{*}$	0.0094
		τ	0.0389

* values are quoted in per annum terms

Table 2
Doubled Immigration Steady State Values

	new	baseline	diff	%
$\bar{K}$	0.9435	0.9379	0.0055	0.59%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8458	0.8380	0.0078	0.93%
$\bar{C}$	0.2777	0.2784	-0.0007	-0.24%
$\bar{I}$	0.2253	0.2180	0.0073	3.36%
$\bar{L}$	0.8015	0.7927	0.0087	1.10%
$\bar{r}^*$	0.1383	0.1379	0.0004	0.32%
$\bar{w}$	0.7071	0.7082	-0.0012	-0.17%
$\bar{T}$	0.0221	0.0242	-0.0021	-8.67%
$\bar{B}$	0.0206	0.0218	-0.0012	-5.63%
$\bar{n}^*$	0.0161	0.0094	0.0066	70.69%
τ	0.0363	0.0389	-0.0025	-6.51%

* values are quoted in per annum terms

Table 3
Skewed Immigration Steady State Values

	new	baseline	diff	%
$\bar{K}$	0.9608	0.9379	0.0229	2.44%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8542	0.8380	0.0162	1.94%
$\bar{C}$	0.2865	0.2784	0.0081	2.91%
$\bar{I}$	0.2154	0.2180	-0.0026	-1.20%
$\bar{L}$	0.8061	0.7927	0.0134	1.69%
$\bar{r}^*$	0.1373	0.1379	-0.0006	-0.46%
$\bar{w}$	0.7100	0.7082	0.0017	0.24%
$\bar{T}$	0.0263	0.0242	0.0021	8.62%
$\bar{B}$	0.0243	0.0218	0.0025	11.34%
$\bar{n}^*$	0.0065	0.0094	-0.0029	-31.09%
τ	0.0424	0.0389	0.0036	9.22%

[*] values are quoted in per annum terms

Table 4
Doubled & Skewed Immigration Steady State Values

	new	baseline	diff	%
$\bar{K}$	0.9885	0.9379	0.0505	5.39%
$\bar{H}$	0.0000	0.0000	0.0000	n/a
$\bar{Y}$	0.8777	0.8380	0.0397	4.74%
$\bar{C}$	0.2935	0.2784	0.0151	5.42%
$\bar{I}$	0.2204	0.2180	0.0024	1.08%
$\bar{L}$	0.8278	0.7927	0.0351	4.43%
$\bar{r}^*$	0.1371	0.1379	-0.0008	-0.57%
$\bar{w}$	0.7104	0.7082	0.0021	0.30%
$\bar{T}$	0.0260	0.0242	0.0018	7.59%
$\bar{B}$	0.0254	0.0218	0.0036	16.46%
$\bar{n}^*$	0.0103	0.0094	0.0009	9.17%
τ	0.0432	0.0389	0.0043	11.18%

* values are quoted in per annum terms

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