SOCIAL SECURITY AND THE INTERACTIONS BETWEEN AGGREGATE & IDIOSYNCRATIC RISK

Daniel Harenberg\textsuperscript{a}  Alexander Ludwig\textsuperscript{b}

\textsuperscript{a}ETH Zurich

\textsuperscript{b}CMR and FiFo, University of Cologne; MEA and Netspar

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Motivation

- Question: Welfare effects of expanding PAYG system?
- Trade-off: Insurance vs. crowding out
- Social security as insurance against
  - Idiosyncratic risk (e.g., Imrohoroglu, et al. (1999))
  - AND
  - Aggregate risk (e.g., Krueger & Kubler (2006))
Interactions

1. Life-cycle interaction ($LCI$)
   - Idiosyncratic wage and aggregate return shocks increase variance of savings
   - Variance of retirement consumption increases
   - Interaction term: $LCI$
   - $LCI$ large, because long time horizon until retirement

2. Counter-cyclical variance of income risk ($CCV$)
   - Idiosyncratic risk higher in downturn than in boom
Approach

- Two-generations model: Main mechanisms
- Quantitative overlapping generations model
- Calibration to the U. S.
- Experiment: Increase social security contributions from 0% to 2%
- Decomposition analysis: Quantify insurance against various sources and interactions
- Robustness and replication of previous literature
Main Results

- Analytically: life-cycle interaction $LCI$
- Positive welfare gains across all calibrations
- Interaction terms ($LCI + CCV$) account for 50-60%
- E.g. baseline calibration (most conservative):
  - Welfare gains in GE: $+1.4\%$
  - Benefits from insurance: $+3.8\%$
  - Losses from crowding out: $-2.4\%$
  - Interactions account for 1/2 of benefits
Two-Generations Model: Households

- Households live 2 periods, consume only when old
- Lifetime utility:
  \[ U_{i,t} = \beta \frac{1}{1 - \theta} c_{i,2,t+1}^{1-\theta} \]
- Budget constraint:
  \[ c_{i,2,t+1} = a'_{i,1,t} (1 + r_{t+1}) + b_{t+1} \]
  \[ a'_{i,1,t} = (1 - \tau) \eta_{i,1,t} w_t \]
Two-Generations Model: Endowments

- Partial equilibrium factor prices:

\[ 1 + r_t = \varrho_t \bar{R} \]
\[ w_t = \zeta_t \bar{w}_t = \zeta_t \bar{w}_{t-1} (1 + g) \]

- PAYG social security:

\[ b_t = \tau w_t \]

- Distribution: jointly log-normal, mean one, independent
Two-Generations Model: Main Result

Proposition

A marginal introduction of social security increases $E_{t-1}U_t$ if

$$(1 + g) \cdot (1 + V)^\theta > \bar{R},$$

where

$$V \equiv \text{var}(\eta_{i,1,t}, \zeta_t \rho_{t+1})$$

$$= \sigma^2_\eta + \sigma^2_\zeta + \sigma^2_\rho + \sigma^2_\zeta \sigma^2_\rho + \sigma^2_\eta \left( \sigma^2_\zeta + \sigma^2_\rho + \sigma^2_\zeta \sigma^2_\rho \right).$$

Daniel Harenberg (ETH Zurich)
Two-Generations Model: Welfare Decomposition

Definition

Consumption equivalent variation, $g_c(\cdot)$:

\[
g_c(IR) = g_c(0) + dg_c(IR) \\
g_c(AR) = g_c(0) + dg_c(AR) \\
g_c(AR, IR) = g_c(0) + dg_c(AR) + dg_c(IR) + dg_c(LCI)
\]

First-order Taylor series approximation of $g_c(AR, IR)$ gives:

\[
g_c(AR, IR) \approx \frac{1 + g}{R} - 1 + \theta \frac{1 + g}{R} AR + \theta \frac{1 + g}{R} IR + \theta \frac{1 + g}{R} LCI
\]
Quantitative Model: Summary

1. Scale-up and extend simple model:
   (a) 70 generations, 1-year periods
   (b) Population growth
   (c) Wage shocks $\Rightarrow$ TFP shocks
   (d) Return shocks $\Rightarrow$ depreciation shocks
   (e) (Auto-)correlation (TFP, depreciation) unrestricted
   (f) Idiosyncratic risk: autocorrelated, CCV
   (g) Deterministic age-income profile
   (h) Epstein-Zin preferences

2. Additional elements:
   (a) Two assets: risk-free bond in addition to risky stock
   (b) Representative firm with capital structure

3. General equilibrium
Competitive recursive equilibrium:

- competitive prices \( \{r, r_f, w\} \), optimal household choices \( \{c, a', \kappa\} \) and firm choices \( \{K, L\} \), market clearing, soc. sec. budget balance \( \{\tau, b\} \), law of motion

Law of motion (Krusell & Smith (1997)):

(i) capital stock, (ii) equity premium

Simulation periods > 80,000

Endogenous grid method (Carroll (2006))

Parallel on 16 cores, computation time 20 - 80 hrs
### Quantitative Model: Baseline Calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Target</th>
<th>(Source)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Working age, retirement age, maximum age</td>
<td>21, 65, 78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Age productivity earnings profiles</td>
<td>(PSID)</td>
<td>({\epsilon_j}_1^J)</td>
<td></td>
</tr>
<tr>
<td>Population growth, (n)</td>
<td>U.S. Social Sec. Adm. (SSA)</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>Technol. growth, (g)</td>
<td>TFP growth (NIPA)</td>
<td>0.018</td>
<td></td>
</tr>
<tr>
<td>Capital share, (\alpha)</td>
<td>wage share (NIPA)</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>Leverage ratio, (d)</td>
<td>U.S. capital structure (Croce (2010))</td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td>Autocorrelation of (\eta)</td>
<td>(Storesletten, et al. (2004))</td>
<td>0.952</td>
<td></td>
</tr>
<tr>
<td>CCV, (\sigma_{\nu,t})</td>
<td>(Storesletten, et al. (2004))</td>
<td>{0.21, 0.13}</td>
<td></td>
</tr>
<tr>
<td>EIS, (\varphi)</td>
<td>exogenous (various)</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>CRRA, (\theta)</td>
<td>exogenous in baseline</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>Discount factor, (\beta)</td>
<td>(K/Y = 2.65) (NIPA)</td>
<td>0.986</td>
<td></td>
</tr>
<tr>
<td>Mean depreciation, (\bar{\delta})</td>
<td>(E(r_f) = 2.3%) (Shiller)</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>Std. depreciation, (\sigma_\delta)</td>
<td>(\sigma:\left( \frac{C_{t+1}}{C_t} \right) = 0.03) (NIPA)</td>
<td>0.08</td>
<td></td>
</tr>
<tr>
<td>Std. TFP shocks, (\sigma_\zeta)</td>
<td>(\sigma(\text{TFP}) = 0.029) (NIPA)</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>Prob((\zeta' = \zeta_i</td>
<td>\zeta = \zeta_i))</td>
<td>autoc(TFP) = 0.88 (NIPA)</td>
<td>0.941</td>
</tr>
<tr>
<td>Prob((\delta' = \delta_i</td>
<td>\zeta' = \zeta_i))</td>
<td>cor(TFP, r) = 0.50 (NIPA, Shiller)</td>
<td>0.885</td>
</tr>
</tbody>
</table>
Experiment: $\tau = 0\% \rightarrow \tau = 2\%$, unanticipated

$g_c$: ex-ante expected CEV of a newborn

<table>
<thead>
<tr>
<th></th>
<th>GE</th>
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</thead>
<tbody>
<tr>
<td>$g_c$</td>
<td>+1.38%</td>
</tr>
<tr>
<td>$\Delta K/K$</td>
<td>-10.42%</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>+0.88%</td>
</tr>
<tr>
<td>$\Delta r_f$</td>
<td>+0.89%</td>
</tr>
<tr>
<td>$\Delta w/w$</td>
<td>-3.47%</td>
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</tbody>
</table>
Results: Partial Equilibrium

- PE: "Small open economy"
- Prices same as in GE, determined by "world"
- No costs of crowding out, isolates benefits

<table>
<thead>
<tr>
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<th>Crowd Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_c$</td>
<td>+1.38%</td>
<td>+3.76%</td>
<td>-2.38%</td>
</tr>
</tbody>
</table>
Results: Decomposition Procedure

- Same PE experiment
- Sequentially "turn off" each risk
- Look at welfare change for each economy
- Recall decomposition of CEV:

\[ g_c(AR, IR, CCV) = g_c(0) + dg_c(AR) + dg_c(IR) + dg_c(LCI) + dg_c(CCV) \]

\[ g_c(AR, IR) = g_c(0) + dg_c(AR) + dg_c(IR) + dg_c(LCI) \]

\[ g_c(AR) = g_c(0) + dg_c(AR) \]

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\[
g_c(AR) = g_c(0) + dg_c(AR)
\]

\[
g_c(IR) = g_c(0) + dg_c(IR)
\]
Results: Decomposition of Welfare Effects

Welfare effects in PE

<table>
<thead>
<tr>
<th>$g_c$</th>
<th>$g_c(0)$</th>
<th>$dg_c(AR)$</th>
<th>$dg_c(IR)$</th>
<th>$dg_c(LCI)$</th>
<th>$dg_c(CCV)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.76%</td>
<td>-0.62%</td>
<td>+1.86%</td>
<td>+0.66%</td>
<td>+1.06%</td>
<td>+0.80%</td>
</tr>
</tbody>
</table>

- Gains from "pure" AR + IR: $dg_c(AR) + dg_c(IR) = 2.52\%$
- Gains from interactions: $dg_c(LCI) + dg_c(CCV) = 1.86\%$

\[
\frac{dg_c(LCI) + dg_c(CCV)}{g_c} = 0.50
\]

\[
\frac{dg_c(LCI)}{dg_c(AR)} = 0.57
\]
Results: Overview of Calibration Strategies

1) IES=0.5
   i) Conservative baseline
   ii) Sharpe ratio
   iii) Equity premium

2) IES=1.5
   i) Conservative baseline
   ii) Sharpe ratio
   iii) Equity premium

3) Alternative calibrations
   i) Contribution rate $\tau = 0.12$
   ii) Mortality risk
   iii) Previous literature
## Results: GE and PE Welfare across Calibrations

### Consumption equivalent variation, $g_c$

<table>
<thead>
<tr>
<th></th>
<th>GE</th>
<th>PE</th>
<th>Crowd Out</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>IES = 0.5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>+1.38%</td>
<td>+3.76%</td>
<td>-2.38%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>+1.54%</td>
<td>+4.56%</td>
<td>-3.02%</td>
</tr>
<tr>
<td>Equity premium</td>
<td>+1.46%</td>
<td>+4.19%</td>
<td>-2.73%</td>
</tr>
<tr>
<td><strong>IES = 1.5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>+1.78%</td>
<td>+2.53%</td>
<td>-0.75%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>+2.05%</td>
<td>+4.28%</td>
<td>-2.23%</td>
</tr>
<tr>
<td>Equity premium</td>
<td>+2.19%</td>
<td>+4.44%</td>
<td>-2.25%</td>
</tr>
</tbody>
</table>
### Results: Decomposition of Welfare Effects

<table>
<thead>
<tr>
<th>Welfare effects in PE</th>
<th>$g_c(0)$</th>
<th>$d g_c(AR)$</th>
<th>$d g_c(IR)$</th>
<th>$d g_c(LCI)$</th>
<th>$d g_c(CCV)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>IES = 0.5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>-0.62%</td>
<td>+1.86%</td>
<td>+0.66%</td>
<td>+1.06%</td>
<td>+0.80%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>-0.62%</td>
<td>+1.52%</td>
<td>+1.13%</td>
<td>+1.23%</td>
<td>+1.30%</td>
</tr>
<tr>
<td>Equity prem.</td>
<td>-0.62%</td>
<td>+1.43%</td>
<td>+0.98%</td>
<td>+0.99%</td>
<td>+1.41%</td>
</tr>
<tr>
<td><strong>IES = 1.5</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>-0.62%</td>
<td>+1.28%</td>
<td>+0.60%</td>
<td>+0.81%</td>
<td>+0.45%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>-0.62%</td>
<td>+1.39%</td>
<td>+1.15%</td>
<td>+1.16%</td>
<td>+1.20%</td>
</tr>
<tr>
<td>Equity prem.</td>
<td>-0.62%</td>
<td>+1.44%</td>
<td>+0.95%</td>
<td>+1.15%</td>
<td>+1.52%</td>
</tr>
</tbody>
</table>
## Results: Welfare Ratios across Calibrations

### Welfare ratios

<table>
<thead>
<tr>
<th></th>
<th>( \frac{dg_c(LCI)}{dg_c(AR)} )</th>
<th>( \frac{dg_c(LCI)+dg_c(CCV)}{dg_c(AR)+dg_c(IR)} )</th>
<th>( \frac{dg_c(LCI)+dg_c(CCV)}{g_c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>IES = 0.5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>0.57</td>
<td>0.74</td>
<td>0.50</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.81</td>
<td>0.95</td>
<td>0.55</td>
</tr>
<tr>
<td>Equity premium</td>
<td>0.69</td>
<td>1.00</td>
<td>0.57</td>
</tr>
<tr>
<td><strong>IES = 1.5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baseline</td>
<td>0.63</td>
<td>0.67</td>
<td>0.50</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.84</td>
<td>0.93</td>
<td>0.55</td>
</tr>
<tr>
<td>Equity premium</td>
<td>0.80</td>
<td>1.12</td>
<td>0.60</td>
</tr>
</tbody>
</table>
Results: Contribution Rate and Mortality Risk

- Contribution rate, $\tau = 0.12$
  - GE welfare: $+1.1\%$
  - Similar pattern, but smaller numbers
    - $\frac{d g_c(LCI) + d g_c(CCV)}{g_c} = 0.28$

- Mortality risk (preliminary)
  - Survival rates from HMD, same expected lifetime
  - Accidental bequests to newborn
  - Need CRRA < 1
  - GE welfare: $+7.3\%$
Results: Consistency with Previous Literature

- Calibration strategy
  - Only idiosyncratic risk
    - GE welfare: $-1.35\%$
  - Only aggregate risk
    - Not yet computed
Conclusion

- Introduction of social security leads to robust welfare gains in GE
- Interaction terms account for at least 1/2 of the benefits
- Social security provides more insurance against aggregate risk than against idiosyncratic income risk
- The larger the social security system, the smaller the welfare gains
- Life-cycle interaction $LCI$ exposed in theoretical model
Outlook: Directions for Future Research

- Companion paper: analytical GE extension
- Endogenous labor
- Optimal size and/or structure of social security
- Government debt / buffer in pension system
Appendix overview I

1. Related Literature
2. Two-Generations Model: GE extension
3. Quantitative Model: Market Structure
4. Quantitative Model: Demographics
5. Quantitative Model: Preferences
6. Quantitative Model: Endowments
7. Quantitative Model: Firms
8. Quantitative Model: Government and social security
9. Quantitative Model: Transformations and definitions
10. Stationary recursive competitive equilibrium
11. Quantitative Model: Household Problem
<table>
<thead>
<tr>
<th>12</th>
<th>Quantitative Model: Laws of Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>Quantitative Model: Mean Shock Equilibrium</td>
</tr>
<tr>
<td>14</td>
<td>Quantitative Model: Transition Matrix</td>
</tr>
<tr>
<td>15</td>
<td>Quantitative Model: Correlation of TFP and Returns</td>
</tr>
<tr>
<td>16</td>
<td>Results: Endogenous Moments</td>
</tr>
<tr>
<td>17</td>
<td>Results: Variance-Covariance Matrix</td>
</tr>
<tr>
<td>18</td>
<td>Results: Life-Cycle Profiles, baseline</td>
</tr>
<tr>
<td>19</td>
<td>Results: Distribution</td>
</tr>
<tr>
<td>20</td>
<td>Results: Different average returns</td>
</tr>
<tr>
<td>21</td>
<td>Results: NC Calibration</td>
</tr>
<tr>
<td>22</td>
<td>Results: PC vs NC welfare</td>
</tr>
</tbody>
</table>
Related Literature

- Quantitative OLG (e.g. Auerbach and Kotlikoff (1987))
- Idiosyncratic risk (e.g. Conesa and Krueger (1999), Imrohoroğlu, Imrohoroğlu, and Joines (1995), Fehr, Habermann, and Kindermann (2008))
- Aggregate risk (e.g. Krueger and Kubler (2006), Bohn (1998))
- Portfolio choice, reasonable equity premium (e.g. Gomes and Michaelides (2008))
- Counter-cyclical variance of income risk / CCV (e.g. Storesletten, Telmer, and Yaron (2007), Constantinides and Duffie (1996), Mankiw (1986))
Two-Generations Model: Welfare Illustration

CEV in Partial Equilibrium, $\theta=3$

- Red line: TR, $\theta=3$
- Dashed black line: AR, $\theta=3$

Daniel Harenberg (ETH Zurich)
Two-Generations Model: GE Extension

- General equilibrium (work in progress)
  - Production economy (Cobb-Douglas)
  - Savings in first period
  - Additional assumptions: log utility, 100% depreciation

- Two additional channels:
  - Precautionary savings
  - Crowding out

- Impact of (interaction of) risks on these two channels
Two-Generations Model: GE Extension

- General equilibrium extension:
  - Savings in first period
  - Idiosyncratic risk in second period (subperiod structure)

- Two additional effects:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Welfare Effect</th>
<th>Interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precautionary savings</td>
<td>positive</td>
<td>positive</td>
</tr>
<tr>
<td>Crowding-out</td>
<td>negative</td>
<td>negative</td>
</tr>
</tbody>
</table>

Daniel Harenberg (ETH Zurich)
Two-Generations Model in GE: Households

- Utility:

\[ E_t U_t = u(c_{i,1,t}) + \beta E_t [u(c_{i,2,t+1})] \]

- Budget constraints:

\[
\begin{align*}
c_{i,1,t} + s_{i,1,t} &= (1 - \tau) w_t \\
c_{i,2,t+1} &= s_{i,1,t} (1 + r_{t+1}) + \lambda \eta_{i,2,t+1} w_{t+1} (1 - \tau) + \\
&+ (1 - \lambda) b_{t+1}
\end{align*}
\]
Two-Generations Model in GE: Government

- Budget constraint:

\[ b_t(1 - \lambda)N_{2,t} = \tau w_t (1 + \lambda) N_{1,t}, \text{ because } N_{2,t} = N_{1,t}, \]

- Therefore:

\[ b_t = \tau w_t \frac{1 + \lambda}{1 - \lambda}. \]
Two-Generations Model in GE: Firms

Profits:

\[
\Pi = \zeta_t F(K_t, \gamma_t L_t) - (\delta + r_t) \rho_t^{-1} K_t - w_t L_t
\]

Production function:

\[
F(K_t, \gamma_t L_t) = K_t^\alpha (\gamma_t L_t)^{1-\alpha}
\]

First-order conditions:

\[
1 + r_t = \alpha K_t^{\alpha-1} \zeta_t \rho_t = \bar{R}_t \zeta_t \rho_t
\]

\[
w_t = (1 - \alpha) \gamma_t K_t^\alpha \zeta_t = \bar{w}_t \zeta_t.
\]
1 Log utility: $u(c) = \ln(c)$

2 100% depreciation: $\bar{\delta} = 1$
**Proposition**

*Equilibrium dynamics in the economy are given by*

\[ k_{t+1} = \frac{1}{(1 + g)(1 + \lambda)} \chi(1 - \tau)(1 - \alpha)\zeta_t k_t^\alpha \]

*where the savings rate \( \chi \) is given by*

\[ \chi \equiv \frac{\beta \bar{E}}{1 + \beta \bar{E}} = \frac{1}{1 + (\beta \bar{E})^{-1}} \]

*and*

\[ \bar{E} \equiv E_t \left[ \frac{1}{1 + \frac{1 - \alpha}{\alpha(1 + \lambda)\zeta_{t+1}} \left( \lambda \eta_{i,2,t+1} + \tau \left( 1 + \lambda (1 - \eta_{i,2,t+1}) \right) \right)} \right] \]
Two-Generations Model in GE: MSE

**Definition**

Mean shock equilibrium (MSE): $\zeta_t = E\zeta_t = 1, \varrho_t = E\varrho_t = 1 \ \forall t$.

Equilibrium dynamics:

$$k_{t+1,ms} = \frac{1}{(1 + g)(1 + \lambda)} \chi(1 - \tau)(1 - \alpha)k_{t,ms}^\alpha$$

**Definition**

Stationary MSE (=stochastic steady state): all variables grow at constant rates: $k_{t,ms} = k_{ms}$ for all $t$.

$$k_{ms} = \left( \frac{1}{(1 + g)(1 + \lambda)} \chi(1 - \tau)(1 - \alpha) \right)^{\frac{1}{1-\alpha}}.$$
Marginal introduction of social security increases ex-ante expected utility in the long-run MSE iff

\[ A + B + C > 0 \]

\[ A \equiv \beta E_{t-1} \left[ \frac{(1-\alpha)}{\alpha} \frac{1}{\zeta_{t+1}} - \frac{(1-\alpha)\lambda}{\alpha(1+\lambda)} \frac{\eta_{i,2,t+1}}{\zeta_{t+1}} - 1 \right] \geq 0 \]

\[ B \equiv \beta \epsilon_{\chi,\tau} \left( \bar{E} - \bar{E} \right) \geq 0 \]

\[ C \equiv -\left( \alpha(1 + \beta) - \beta(1 - \alpha)\bar{E} \right) \frac{1}{1 - \alpha} \frac{(1 - \epsilon_{\chi,\tau})}{\geq 0} < 0 \]
Two-Generations Model in GE: Term $A$

The graph illustrates the relationship between aggregate risk and term $A$ for two scenarios: zero idiosyncratic risk and full idiosyncratic risk. The graph shows a positive correlation between aggregate risk and term $A$ for both scenarios, with the full idiosyncratic risk scenario exhibiting a steeper increase compared to the zero idiosyncratic risk scenario.
Two-Generations Model in GE: Terms $B&C$

![Graph showing the relationship between aggregate risk and terms $B$ and $C$. The graph for term $B$ shows an increasing trend with red and black markers representing zero and full idiosyncratic risk, respectively. The graph for term $C$ shows a decreasing trend with similar markers.]

Daniel Harenberg (ETH Zurich)
Discrete time $t = 0, \ldots, \infty$

Aggregate shock $z_t$: Markov chain with $\pi(Z_{t+1} \mid Z_t)$

Event tree $z^t = (z_0, z_1, \ldots, z_t)$

Incomplete markets

- Bond: one-period risk-free at known interest rate $r_{t+1}^f$
- Stock: risky return $r_{t+1}$

Natural borrowing limit
Quantitative Model: Demographics

- $J$ overlapping generations, indexed by $j = 1, \ldots, J$
- Retirement age $j r$
- Survival probabilities $s_{j+1}$
- Accidental bequests are burned
- Population grows at rate $n$
- Continuum of agents in each generation
- Intragenerational heterogeneity denoted by $i$
Quantitative Model: Preferences

- Epstein-Zin preferences:

\[ U_{i,j,t} = \left[ \frac{1-\theta}{\gamma} c_{i,j,t} + \beta s_{j+1} \left( \mathbb{E} \left[ U_{i,j+1,t+1}^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \]

- \( \theta \): Coefficient of relative risk-aversion
- \( \varphi \): Elasticity of intertemporal substitution
- \( \gamma = \frac{1-\theta}{1-\frac{1}{\varphi}} \)
Quantitative Model: Endowments

- Dynamic budget constraint:
  \[ a'_{i,j,t} + c_{i,j,t} = a_{i,j,t}(1 + r^f_t + \kappa_{i,j-1,t-1}(r_t - r^f_t)) + y_{i,j,t} \]
  with \( a'_{i,J,t} \geq 0 \)

- Income:
  \[ y_{i,j,t} = \begin{cases} 
  (1 - \tau) \eta_{i,j,t} w_t \epsilon_j & \text{for } j < j_{ret} \\
  b_t & \text{for } j \geq j_{ret} 
  \end{cases} \]

- Idiosyncratic stochastic component:
  \[ \ln \eta_{i,j,t} = \rho \ln \eta_{i,j-1,t-1} + \nu_{i,t}, \quad \sigma^2_\nu(contr) > \sigma^2_\nu(expans) \]
Quantitative Model: Firms

- Neoclassical production:

\[ Y_t = F(\zeta_t, K_t, L_t) = \zeta_t K_t^{\alpha} (\gamma_t L_t)^{1-\alpha} \]

- Wage rate:

\[ w_t = \zeta_t (1 - \alpha) K_t^\alpha (1 + g) \gamma_{t-1} \]

- Net return on capital:

\[ r_t^k = \zeta_t^\alpha K_t^{\alpha-1} - \delta_t \]

- Leveraged stock return:

\[ r_t = r_t^k (1 + d) - dr_t^f \]
Quantitative Model: Government and social security

- Government collects accidental bequests and burns them
- PAYG budget constraint:

\[ \tau_t w_t L_t = Ret_t \int P_{i,j,t} d\Phi \]

- For today:
  - fixed contribution rate: \( \tau_t = \tau \)
  - lump-sum benefits: \( P_{i,j,t} = P_t \)
- Experiment: single, unanticipated increase in \( \tau \)
Rewrite in terms of cash at hand, $x$

$$x = a(1 + r^f + \kappa(r - r^f)) + y$$

Denote measure over agents by $\Phi_t(j, x, \eta)$

State space for each agent: $S = (j, x, \eta, z, \Phi)$
Stationary recursive competitive equilibrium

- Price functions \( \{ r(\Phi, z), r^f(\Phi, z), w(\Phi, z) \} \)
- Policy functions \( c(S), a'(S), \kappa(S) \) that maximize the household’s utility for given \( \{ r, r^f, w, \tau, b \} \)
- Firm choice \( k \) that maximizes profits for given \( \{ r, r^f, w \} \)
- Govt policies \( \tau(\Phi, z), b(\Phi, z) \) implying budget balance
- Market clearing, in particular:

\[
k'(\Phi', z') = \int a'(S) \, d\Phi(j, x, \eta)
\]

\[
B'(\Phi', z') = \int (1 - \kappa(S)) \, a'(S) \, d\Phi(j, x, \eta)
\]

- A law of motion \( \Phi' = H(\Phi, z, z') \) consistent with policies
Quantitative Model: Household problem

- Euler equations

\[
c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta} \left( \mathbb{E} \left[ u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \cdots
\]

\[
\cdot \mathbb{E} \left[ u(j+1, \cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \tilde{R}' \right] = 0
\]

\[
\mathbb{E} \left[ u(j+1, \cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} (r' - r^{f'}) \right] = 0
\]

- Endogenous grid method (Carroll 2006) applied to portfolio choice
  - Avoid collinear problem of jointly finding \( \{a', \kappa\} \)
  - Reduce 2-dimensional optimization to 2 sequential steps: first solve for \( \kappa \), then for \( c \)
Quantitative Model: Laws of motion

- Problem: agents need measure $\Phi$ to forecast prices
- Krusell and Smith (1998): approximate $\Phi' = H(\Phi, z, z')$ by low-dimensional object
- Our approximation is

$$ (k', \mu) = \hat{H}(k, k^2, z) $$

where $\mu = \mathbb{E}r' - r^{ff}$, the expected equity premium
- To find $\hat{H}$, need to simulate and update until convergence
- Fit: $R^2 = 0.9999$
Quantitative Model: Mean shock equilibrium

- Auxiliary general equilibrium
- Degenerate laws of motion: \((k', \mu') = k, \mu\)
- Solve household problem for all \(z\)
- Instead of simulating, set \(z = \bar{z}\), with \(\zeta(\bar{z}) = \mathbb{E}\zeta\) and \(\delta(\bar{z}) = \mathbb{E}\delta\)
- Find fixed point: \((k', \mu')\) is generated by \(a'(S, k, \mu)\) and \(\kappa'(S, k, \mu)\)
- Can use \(k_{ms}, \mu_{ms}, \Phi_{ms}\) as initial guesses for KS method
Quantitative Model: Transition Matrix $\pi(z'|z)$

- $\pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$
- $\pi^\delta = \pi(\delta' = \delta_0 + \bar{\delta} \mid \zeta' = 1 - \bar{\zeta}) = \pi(\delta' = \delta_0 - \bar{\delta} \mid \zeta' = 1 + \bar{\zeta})$
- both symmetric

$$
\pi^Z = \begin{bmatrix}
\pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot \pi^\delta \\
\pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot \pi^\delta \\
(1 - \pi^\zeta) \cdot \pi^\delta & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & \pi^\zeta \cdot (1 - \pi^\delta) & \pi^\zeta \cdot \pi^\delta \\
(1 - \pi^\zeta) \cdot \pi^\delta & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & \pi^\zeta \cdot (1 - \pi^\delta) & \pi^\zeta \cdot \pi^\delta 
\end{bmatrix}
$$

- STY: $\pi^\delta = 1$
- GM: $\pi^\delta = 0.5$
- Our paper: $\pi^\delta = 0.7$
Quantitative Model: Correlation of TFP and Returns

Daniel Harenberg (ETH Zurich)
Social Security and Risk Interactions
Logan, May 25, 2013 52 / 24
Results: Distribution over Age and Cash-at-Hand

\[ \Phi_{FD1, K/g}(x) \]

\[ \Phi_{FD2, K/g}(x) \]

\[ \Phi_{FD3, K/g}(x) \]

\[ \Phi_{FD4, K/g}(x) \]
Results: Distribution over Age and Cash-at-Hand

\[ \Phi_{FD_1K,ge}(x,j), \text{ for } \Phi > 1 \times 10^{-05} \]
Results: Economy without Aggregate Risk

- Only one asset
- Empirical average asset return (Siegel (2002)): 4.2%
- Model average asset returns

<table>
<thead>
<tr>
<th>Equity premium calibration</th>
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</thead>
<tbody>
<tr>
<td>Median portfolio return</td>
</tr>
<tr>
<td>E(mpk)</td>
</tr>
<tr>
<td>Capital-structure weighted</td>
</tr>
<tr>
<td>average of $E(r)$ and $E(r_f)$</td>
</tr>
</tbody>
</table>

- Comparable and consistent results
### Results: Different average returns

#### Equity premium calibration with mortality

<table>
<thead>
<tr>
<th></th>
<th>Median((pfr))</th>
<th>(E(mpk))</th>
<th>(mpk (E(r), E(r_f)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g(0, IR))</td>
<td>1.360%</td>
<td>-0.099%</td>
<td>-0.438%</td>
</tr>
<tr>
<td>(g(0, 0))</td>
<td>-0.102%</td>
<td>-0.912%</td>
<td>-1.101%</td>
</tr>
<tr>
<td>(dg(AR))</td>
<td>1.066%</td>
<td>1.876%</td>
<td>2.066%</td>
</tr>
<tr>
<td>(dg(IR))</td>
<td>1.461%</td>
<td>0.813%</td>
<td>0.664%</td>
</tr>
<tr>
<td>(dg(LCI))</td>
<td>0.949%</td>
<td>1.598%</td>
<td>1.747%</td>
</tr>
<tr>
<td>(dg(LCI)/dg(AR))</td>
<td>0.891</td>
<td>0.851</td>
<td>0.846</td>
</tr>
<tr>
<td>(dg(AR) + dg(IR))</td>
<td>2.528%</td>
<td>2.690%</td>
<td>2.730%</td>
</tr>
<tr>
<td>(dg(LCI) + dg(CCV))</td>
<td>2.608%</td>
<td>3.256%</td>
<td>3.406%</td>
</tr>
</tbody>
</table>

\[
\text{mpk} (E(r), E(r_f)) = \frac{E(r) + \bar{d} \cdot E(r_f)}{1 + \bar{d}}
\]
## Results: NC Calibration

<table>
<thead>
<tr>
<th></th>
<th>PC</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Target</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corr. TFP, R, (\text{corr}(\zeta_t, R_t))</td>
<td>0.50</td>
<td>-0.08</td>
</tr>
<tr>
<td><strong>Main parameter</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cond. prob. depr. shocks, (\pi^\delta)</td>
<td>0.86</td>
<td>0.435</td>
</tr>
<tr>
<td><strong>Adjustments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discount factor, (\beta)</td>
<td>0.97</td>
<td>0.96</td>
</tr>
<tr>
<td>Relative risk aversion, (\theta)</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Small adjustments in (\tilde{\delta}, \sigma_\delta)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Endogenous moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corr. w, R, (\text{corr}(w_t, R_t))</td>
<td>0.306</td>
<td>-0.33</td>
</tr>
</tbody>
</table>
## Results: PC vs NC welfare

<table>
<thead>
<tr>
<th>Welfare gains</th>
<th>PC</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_c(0, 0)$</td>
<td>-2.00%</td>
<td>-2.00%</td>
</tr>
<tr>
<td>$dg_c(AR)$</td>
<td>3.26%</td>
<td>2.18%</td>
</tr>
<tr>
<td>$dg_c(IR)$</td>
<td>1.00%</td>
<td>1.04%</td>
</tr>
<tr>
<td>$dg_c(LCI)$</td>
<td>1.66%</td>
<td>0.14%</td>
</tr>
<tr>
<td>$dg_c(CCV)$</td>
<td>1.77%</td>
<td>0.47%</td>
</tr>
</tbody>
</table>


